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ON BIPARTITIONAL FUNCTIONS

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Bipartitional functions are arithmetical functions of two partitions of the same number, and arise primarily in the theory of the symmetric function generating functions.

Analytical methods of evaluating the partitional functions and of studying them in relation to the theory of distributions are largely due to Macmahon (1915). The use of partitional notation has rendered his methods distinctly simpler than those of his predecessors, but, simplified as they are, his methods do not make the practical evaluation of these functions particularly expeditious. If his methods are actually put into practice, it is found that they become increasingly laborious and impracticable with high-order symmetric functions. An excellent example of the difficulties encountered in the use of algebraic methods, especially those involving the action of differential operators, is to be found in the enumeration of the 5×5 and 6×6 Latin Squares (Fisher and Yates 1934). In this connexion it is shown by Fisher and Yates that the direct enumeration by trial is a much simpler approach than the development of the differential operators of Macmahon's algebraic solution.

Bipartitional functions derive much of their importance on account of the combinatorial problems of which they supply solutions. They have, consequently, important applications in applied statistics, as, for example, in the derivation of the moments, and the product moments of moment statistics (Fisher 1930), in the enumeration of different samples of a given size drawn from a finite population, and so on. It is, therefore, desirable that the arithmetic implied in the algebraic methods of evaluating the partitional functions should be clearly and systematically set out in standard form for ready evaluation. Professor R. A. Fisher suggested to me to take up for a systematic study the problem of evaluating the bipartitional functions, and of formulating their relations to distributions *in plano*, and to the combinatorial problems of which they supply solutions. It is the purpose of this paper to put forward these relationships in a comparable manner, so as to give a comprehensive view of at least some aspects of their properties.

The symmetric functions discussed by Macmahon are (1) the monomial symmetric functions, or the g-functions, in the notation of this paper; (2) the elementary symmetric functions called the a-functions; (3) the homogeneous product sums called the h-functions; and (4) the sums of powers called the s-functions of the given quantities $\alpha_1, \alpha_2, \alpha_3, \ldots$.

In the partitional notation, corresponding to any partition P, $=(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots)$, of the partible number w, where $\Sigma p\pi = w$ and $\Sigma\pi=\rho$, the monomial symmetric function of the α's given by

$$G(P) = \Sigma\alpha_1^{p_1}\alpha_2^{p_1}\ldots\alpha_{\pi_1}^{p_1}\alpha_{\pi_1+1}^{p_2}\ldots\alpha_{\pi_1+\pi_2}^{p_2}$$

is denoted by $G(P)$; the a-product $a_{p_1}^{\pi_1}a_{p_2}^{\pi_2}\ldots$ may be written as $A(P)$; the h-product $h_{p_1}^{\pi_1}h_{p_2}^{\pi_2}\ldots$ as $H(P)$, and the s-product $s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}$ as $S(P)$.

The simple symmetric functions a_p, h_p and s_p are expressed in terms of monomial functions as follows

$$a_p = G(1^p), \quad s_p = G(p), \quad h_p = \Sigma G(p),$$

the summation being taken over all partitions of the number p.

Any of these functions may be expressed as products of the same weight in one of the other symbols by the use of the identity

$$\begin{aligned} & s_1x+s_2\frac{x^2}{2}+s_3\frac{x^3}{3}+s_4\frac{x^4}{4}+\ldots \\ & \equiv -\log(1-a_1x+a_2x^2-a_3x^3+a_4x^4-\ldots) \\ & \equiv \quad \log(1+h_1x+h_2x^2+h_3x^3+h_4x^4+\ldots) \\ & \equiv -\Sigma\log(1-\alpha x), \end{aligned}$$

the summation being over an indefinite number of the dummy variables α.

It is now apparent that any expression such as $A(P)$ can be expanded as a linear function of $S(Q)$, where Q is any partition of the same number as is P. The coefficients of this expansion will be arithmetical functions merely of the two partitions P and Q, and we designate the coefficients by $As(P, Q)$ defined by the identity

$$A(P) \equiv \Sigma As(P,Q).S(Q),$$

the summation being taken over all partitions of the same number.

The expansions of each of the four expressions A, S, H and G, each in terms of the three others, then serve to define twelve primary bipartitional functions. From their definition it is evident that they yield (a) six identities of the form

$$\sum_Q As(P,Q)\,Sa(Q,P) \equiv 1,$$

$$\sum_Q As(P,Q)\,Sa(Q,R) \equiv 0, \quad R \neq P,$$

in addition (b) to twelve identities of the form

$$Ga(P,R) \equiv \sum_Q Gh(P,Q).Ha(Q,R).$$

For partitions of very small numbers, which have few partitions, these identities may be used to obtain any one bipartitional function from certain others. The number of partitions of 13 is, however, 100, so that more expeditious methods are clearly desirable. Nevertheless, the principle of this method may be used with advantage in the evaluation of *Ga* and *Gh*, for partitions of fairly large numbers.

The methods of evaluating the first six partitional functions by differential operators and the last six by algebraic relations are discussed by Macmahon, and in particular the values of *Ag*, *Ga*, *Hg* and *Gh* for all pairs of partitions of weight less than, or equal to, six are tabulated. In what follows we shall give the tables of the twelve partitional functions for weights less than, or equal to, eight, and discuss the arithmetical methods of evaluating the high-order partitional functions. We shall also give illustrations of their combinatorial interpretation, and note their summation properties.

Of the twelve partitional functions it is found necessary to tabulate and discuss only nine, namely *Ag*, *Ga*; *Hg*, *Gh*; *Sg*, *Gs*; *As*, *Sa* and *Ah*. The partitional functions *As*, *Sa* and *Ah* are related to *Hs*, *Sh* and *Ha* by the following simple relations

$$Hs(P,Q) \equiv (-1)^{w+\sigma} As(P,Q),$$
$$Sh(P,Q) \equiv (-1)^{w+\rho} Sa(P,Q),$$
$$Ha(P,Q) \equiv Ah(P,Q),$$

where w stands for the partible number and ρ, σ are the numbers of parts in P and Q.

Definitions. The terminology concerning separations used in this paper is taken from Macmahon's work (1915). Some of the less familiar terms are defined below:

Separation: When all the parts of a set of partitions, say $(p_1 p_2)$, $(p_3 p_4 p_5)$, $(p_6 p_7)$, ..., are assembled to form a single partition $(p_1 p_2 \ldots p_7)$, then $(p_1 p_2)(p_3 p_4 p_5)(p_6 p_7)$ is said to form a separation of the separable partition $(p_1 p_2 \ldots p_7)$.

Separates: The partitions $(p_1 p_2)$, $(p_3 p_4 p_5)$, $(p_6 p_7)$ are called the separates of the separation $(p_1 p_2)(p_3 p_4 p_5)(p_6 p_7)$.

Specification: If the successive weights of the separates be $w_1 w_2 w_3 \ldots$, then the separation is said to have a specification $(w_1 w_2 w_3 \ldots)$.

The bipartitional function $Sa(P,Q) \equiv (-)^{w+\rho} Sh(P,Q)$

(*a*) *Algebraic definition:*

$$s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots = \Sigma Sa(P,Q)\, a_{q_1}^{\chi_1} a_{q_2}^{\chi_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$, and the summation Σ is taken over all the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$(-1)^{w-\sigma} Sa(P,Q)$ is the number of ways of selecting ρ objects, one from each of the lowest rows in ρ columns of the two-way distributions so that

(i) The column totals form the partition P.

(ii) The row totals form the partition Q.

(iii) No column with lower total precedes any column with higher total.

(iv) Each row is represented in only one column.

(v) No row represented in a later column precedes any row represented in an earlier column.

(vi) Rows represented in the same column are arranged in any order, equal rows being undistinguished.

Example. To evaluate $Sa(3^22,\ 2^21^4)$.

The two-way distributions satisfying the six conditions are:

2	2	.	.	2	2	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1	1	.	.	2	2	.	.	1	1	.	.
1	1	.	.	1	1	.	.	2	2	.	.	2	2	.	.	1	1	.	.	1	1	.	.	1	1	.	.	2	2	.	.
2	.	2	.	1	.	1	.	2	.	2	.	1	.	1	.	1	1	.	.	1	1	.	.	1	.	1	.	1	.	1	.
1	.	1	.	2	.	2	.	1	.	1	.	2	.	2	.	1	.	2	.	1	.	1	.	1	.	1	.	1	.	1	.
1	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1	2	.	1	.	1	.	2	.	1	.	1	.	1	.	1	.
1	.	.	1	1	.	.	1	1	.	.	1	2	.	.	1	2	.	.	2	2	.	.	2	2	.	.	2	2	.	.	2
	3	3	2		3	3	2		3	3	2		3	3	2		3	3	2		3	3	2		3	3	2		3	3	2
		1				2				2				4				2				4				2				4	

The numbers of ways of selecting three objects, one from each of the lowest rows in three columns of each of the eight two-way distributions, are written just below the distributions. Their sum is 21 which is the required arithmetical function.

(*c*) *Practical evaluation for larger partitions.*

All the separations of Q whose specification is P are listed. If, in a separation, any separate of weight p is composed of v parts of which sets of $\pi'_1, \pi'_2, \ldots$ are equal, then the separate is scored with the number $\dfrac{(v-1)!}{\pi'_1!\,\pi'_2!\ldots}p$. The product of these scores for all separates in a separation multiplied by the numbers of permutations among separates of equal weight is then assigned to each separation. The sum of these for all separations gives $(-1)^{w+\sigma}Sa(P, Q)$.

Example. To evaluate $Sa(64^23^2,\ 432^31^7)$.

The partition (432^31^7) has twenty-five separations with specification (64^23^2). Each separation is scored with a number written beside it.

$(42)\ (31)\ (2^2)\ (1^3)^2$	96
$(321)\ (4)\ (2^2)\ (1^3)^2$	192
$(2^3)\ (4)\ (31)\ (1^3)^2$	64
$(42)\ (31)\ (21^2)\ (1^3)\ (21)$	1152
$(321)\ (4)\ (21^2)\ (1^3)\ (21)$	2304
$(41^2)\ (31)\ (2^2)\ (1^3)\ (21)$	576
$(31^3)\ (4)\ (2^2)\ (1^3)\ (21)$	576
$(2^21^2)\ (4)\ (31)\ (1^3)\ (21)$	1728
$(41^2)\ (31)\ (21^2)\ (21)^2$	1728
$(31^3)\ (4)\ (21^2)\ (21)^2$	1728
$(21^4)\ (4)\ (31)\ (21)^2$	1728
$(42)\ (31)\ (1^4)\ (21)^2$	432
Subtotal	12304

Subtotal	12304
$(321)\ (4)\ (1^4)\ (21)^2$	864
$(42)\ (21^2)\ (1^4)\ (21)\ (3)$	864
$(41^2)\ (2^2)\ (1^4)\ (21)\ (3)$	432
$(2^21^2)\ (4)\ (1^4)\ (21)\ (3)$	1296
$(41^2)\ (21^2)^2\ (21)\ (3)$	1728
$(21^4)\ (4)\ (21^2)\ (21)\ (3)$	3456
$(1^6)\ (4)\ (2^2)\ (21)\ (3)$	288
$(42)\ (2^2)\ (1^4)\ (3)\ (1^3)$	144
$(2^3)\ (4)\ (1^4)\ (3)\ (1^3)$	96
$(42)\ (21^2)^2\ (3)\ (1^3)$	576
$(41^2)\ (2^2)\ (21^2)\ (3)\ (1^3)$	576
$(2^21^2)\ (4)\ (21^2)\ (3)\ (1^3)$	1728
$(21^4)\ (4)\ (2^2)\ (3)\ (1^3)$	576
Total	24928

(*d*) *Combinatorial problem.*

Five sets of six red, four white, four blue, three green and three pink numbered cards are divided into twelve packs of four, three, two, two, two, one, one, one, one, one, one, one cards so that each pack contains cards of only one colour. The packs belonging to each set are then grouped together and from the uppermost pack of each set a card is drawn. In how many ways can the operation be carried out if in grouping packs of the same set equal packs are undistinguished?

(*e*) *Summation properties.*

$\sum_{Q/\sigma} Sa(P, Q) =$ coefficient of x^{σ} in

$$(-1)^w\{(1-x)^w - \sum_i (1-x)^{w-p_i} + \sum_i \sum_j (1-x)^{w-p_i-p_j} - \sum_i \sum_j \sum_k (1-x)^{w-p_i-p_j-p_k} \ldots\}$$

$$= (-1)^{w+\sigma}\{{}^wC_\sigma - \sum_i {}^{w-p_i}C_\sigma + \sum_i \sum_j {}^{w-p_i-p_j}C_\sigma \ldots\},$$

where $\sum_{Q/\sigma}$ denotes summation over all partitions of Q of σ parts; $w =$ total weight $= \Sigma p_i w_i = \Sigma q_i x_i$ and ${}^wC_\sigma = \dfrac{w!}{\sigma!(w-\sigma)!}$.

THE BIPARTITIONAL FUNCTION $As(P, Q) \equiv (-)^{w+\sigma} Hs(P, Q)$

(*a*) *Algebraic definition:*

$$a_{p_1}^{\pi_1} a_{p_2}^{\pi_2} \ldots = \Sigma As(P, Q)\, s_{q_1}^{x_1} s_{q_2}^{x_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{x_1} q_2^{x_2} \ldots)$ and the summation Σ is taken over all the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$(-1)^{w-\sigma} \dfrac{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \ldots}{\{(q_1-1)!\}^{x_1} \{(q_2-1)!\}^{x_2} \ldots} As(P, Q)$ is the number of ways of distributing different objects contained by the columns of the two-way distributions so that

(i) The column totals form the partition P.
(ii) The row totals form the partition Q.
(iii) No column with lower total precedes any column with higher total.
(iv) Each row is represented in only one column.
(v) No row represented in a later column precedes any row represented in an earlier column.
(vi) Among rows represented in the same column, no row with lower total precedes any row with higher total, equal rows being undistinguished.

TABLES OF $Sa(P, Q)$. $s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots$ IN TERMS OF $a_{q_1}^{x_1} a_{q_2}^{x_2} \ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1²)
(2)	−2	1
(1²)	.	1

P \ Q	(3)	(21)	(1³)
(3)	3	−3	1
(21)	.	−2	1
(1³)	.	.	1

P \ Q	(4)	(31)	(2²)	(21²)	(1⁴)
(4)	−4	4	2	−4	1
(31)	.	3	.	−3	1
(2²)	.	.	4	−4	1
(21²)	.	.	.	−2	1
(1⁴)	.	.	.	.	1

P \ Q	(5)	(41)	(32)	(31²)	(2²1)	(21³)	(1⁵)
(5)	5	−5	−5	5	5	−5	1
(41)	.	−4	.	4	2	−4	1
(32)	.	.	−6	3	6	−5	1
(31²)	.	.	.	3	.	−3	1
(2²1)	.	.	.	.	4	−4	1
(21³)	.	.	.	.	.	−2	1
(1⁵)	.	.	.	.	.	.	1

P \ Q	(6)	(51)	(42)	(3²)	(41²)	(321)	(2³)	(31³)	(2²1²)	(21⁴)	(1⁶)
(6)	−6	6	6	3	−6	−12	−2	6	9	−6	1
(51)	.	5	.	.	−5	−5	.	5	5	−5	1
(42)	.	.	8	.	−4	−8	−4	4	10	−6	1
(3²)	.	.	.	9	.	−18	.	6	9	−6	1
(41²)	.	.	.	.	−4	.	.	4	2	−4	1
(321)	.	.	.	.	.	−6	.	3	6	−5	1
(2³)	.	.	.	.	.	.	−8	.	12	−6	1
(31³)	.	.	.	.	.	.	.	3	.	−3	1
(2²1²)	.	.	.	.	.	.	.	.	4	−4	1
(21⁴)	.	.	.	.	.	.	.	.	.	−2	1
(1⁶)	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(7)	(61)	(52)	(43)	(51²)	(421)	(3²1)	(32²)	(41³)	(321²)	(2³1)	(31⁴)	(2²1³)	(21⁵)	(1⁷)
(7)	7	−7	−7	−7	7	14	7	7	−7	−21	−7	7	14	−7	1
(61)	.	−6	.	.	6	6	3	.	−6	−12	−2	6	9	−6	1
(52)	.	.	−10	.	5	10	.	10	−5	−15	−10	5	15	−7	1
(43)	.	.	.	−12	.	12	12	6	−4	−24	−6	7	14	−7	1
(51²)	.	.	.	.	5	.	.	.	−5	−5	.	5	5	−5	1
(421)	.	.	.	.	.	8	.	.	−4	−8	−4	4	10	−6	1
(3²1)	.	.	.	.	.	.	9	.	.	−18	.	6	9	−6	1
(32²)	.	.	.	.	.	.	.	12	.	−12	−12	3	16	−7	1
(41³)	.	.	.	.	.	.	.	.	−4	.	.	4	2	−4	1
(321²)	.	.	.	.	.	.	.	.	.	−6	.	3	6	−5	1
(2³1)	.	.	.	.	.	.	.	.	.	.	−8	.	12	−6	1
(31⁴)	.	.	.	.	.	.	.	.	.	.	.	3	.	−3	1
(2²1³)	.	.	.	.	.	.	.	.	.	.	.	.	4	−4	1
(21⁵)	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	1
(1⁷)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(8)	(71)	(62)	(53)	(4²)	(61²)	(521)	(42²)	(431)	(3²2)	(51³)	(421²)	(3²1²)	(32²1)	(2⁴)	(41⁴)	(321³)	(2³1²)	(31⁵)	(2²1⁴)	(21⁶)	(1⁸)
(8)	−8	8	8	8	4	−8	−16	−8	−16	−8	8	24	12	24	2	−8	−32	−16	8	20	−8	1
(71)	.	7	.	.	.	−7	−7	.	−7	.	7	14	7	7	.	−7	−21	−7	7	14	−7	1
(62)	.	.	12	.	.	−6	−12	−12	.	−6	6	18	3	24	4	−6	−24	−20	6	21	−8	1
(53)	.	.	.	15	.	.	−15	.	−15	−15	5	15	15	30	.	−5	−35	−15	8	20	−8	1
(4²)	.	.	.	.	16	.	.	−16	−32	.	.	32	16	16	4	−8	−32	−16	8	20	−8	1
(61²)	.	.	.	.	.	−6	.	.	.	.	6	6	3	.	.	−6	−12	−2	6	9	−6	1
(521)	.	.	.	.	.	.	−10	.	.	.	5	10	.	10	.	−5	−15	−10	5	15	−7	1
(42²)	.	.	.	.	.	.	.	−16	.	.	.	16	.	16	8	−4	−16	−24	4	22	−8	1
(431)	.	.	.	.	.	.	.	.	−12	.	.	12	12	6	.	−4	−24	−6	7	14	−7	1
(3²2)	.	.	.	.	.	.	.	.	.	−18	.	.	9	36	.	.	−30	−18	6	21	−8	1
(51³)	.	.	.	.	.	.	.	.	.	.	5	.	.	.	.	−5	−5	.	5	5	−5	1
(421²)	.	.	.	.	.	.	.	.	.	.	.	8	.	.	.	−4	−8	−4	4	10	−6	1
(3²1²)	.	.	.	.	.	.	.	.	.	.	.	.	9	.	.	.	−18	.	6	9	−6	1
(32²1)	.	.	.	.	.	.	.	.	.	.	.	.	.	12	.	.	−12	−12	3	16	−7	1
(2⁴)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	16	.	.	−32	.	24	−8	1
(41⁴)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−4	.	.	4	2	−4	1
(321³)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−6	.	3	6	−5	1
(2³1²)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−8	.	12	−6	1
(31⁵)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	−3	1
(2²1⁴)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4	−4	1
(21⁶)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−2	1
(1⁸)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Example. To evaluate $(-1)^2\dfrac{(3!)^2\,(2!)}{(1!)^2\,(0!)^4}As(3^22,\,2^21^4)$.

The two-way distributions satisfying the six conditions of the definition are

$$\begin{array}{c|ccc} 2 & 2 & . & . \\ 1 & 1 & . & . \\ 2 & . & 2 & . \\ 1 & . & 1 & . \\ 1 & . & . & 1 \\ 1 & . & . & 1 \\ \hline & 3 & 3 & 2 \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & . & . \\ 1 & 1 & . & . \\ 1 & . & 1 & . \\ 1 & . & 1 & . \\ 1 & . & 1 & . \\ 2 & . & . & 2 \\ \hline & 3 & 3 & 2 \end{array} \qquad \begin{array}{c|ccc} 1 & 1 & . & . \\ 1 & 1 & . & . \\ 1 & 1 & . & . \\ 2 & . & 2 & . \\ 1 & . & 1 & . \\ 2 & . & . & 2 \\ \hline & 3 & 3 & 2 \end{array}$$

$$9 \qquad\qquad\qquad 3 \qquad\qquad\qquad 3$$

The numbers of ways of distributing different objects contained by the columns of each of the three two-way distributions are written just below the distributions. Their sum is 15, which is the required answer.

(*c*) *Practical evaluation for larger partitions.*

All the separations of Q whose specification is P are listed. If, in a separation, any separate of weight p is, say, $(q_i^{\pi_1'}\,q_j^{\pi_2'}\ldots)$, the separate is scored with the number

$$\frac{p!}{(q_i!\,q_j!\,\ldots)\,(\pi_1'!\,\pi_2'!\,\ldots)}.$$

The product of these scores for all separates in a separation multiplied by the numbers of permutations among separates of equal weight is then assigned to each separation. The sum of these scores for all separations gives

$$(-1)^{w-\sigma}\frac{(p_1!)^{\pi_1}\,(p_2!)^{\pi_2}\ldots}{\{(q_1-1)!\}^{x_1}\,\{(q_2-1)!\}^{x_2}\ldots}As(P,\,Q).$$

Example. To evaluate $(-1)^7\dfrac{(5!)^2\,(4!)\,(3!)^2}{(2!)^2\,(1!)^3\,(0!)^8}As(5^243^2,\,3^22^31^8)$.

The partition $(3^22^31^8)$ has twenty-nine separations with specification (5^243^2). The score of each separation is written beside it.

$(32)\,(2^21)\,(31)\,(1^3)^2$	1200
$(32)\,(31^2)\,(2^2)\,(1^3)^2$	600
$(32)^2\,(21^2)\,(1^3)^2$	600
$(32)^2\,(1^4)\,(1^3)\,(21)$	600
$(32)\,(31^2)\,(21^2)\,(1^3)\,(21)$	7200
$(31^2)^2\,(2^2)\,(1^3)\,(21)$	1800
$(31^2)\,(2^21)\,(31)\,(1^3)\,(21)$	7200
$(32)\,(21^3)\,(31)\,(1^3)\,(21)$	4800
$(32)\,(2^21)\,(1^4)\,(1^3)\,(3)$	600
$(31^2)\,(2^21)\,(21^2)\,(1^3)\,(3)$	3600
$(32)\,(21^3)\,(21^2)\,(1^3)\,(3)$	2400
$(31^2)\,(21^3)\,(2^2)\,(1^3)\,(3)$	1200
$(32)\,(1^5)\,(2^2)\,(1^3)\,(3)$	120
$(21^3)\,(2^21)\,(31)\,(1^3)\,(3)$	2400
Subtotal	34320

Subtotal	34320
$(31^2)\,(32)\,(1^4)\,(21)^2$	1800
$(31^2)^2\,(21^2)\,(21)^2$	5400
$(1^5)\,(32)\,(31)\,(21)^2$	720
$(21^3)\,(31^2)\,(31)\,(21)^2$	7200
$(32)\,(21^3)\,(1^4)\,(21)\,(3)$	1200
$(2^21)\,(31^2)\,(1^4)\,(21)\,(3)$	1800
$(1^5)\,(32)\,(21^2)\,(21)\,(3)$	720
$(31^2)\,(21^3)\,(21^2)\,(21)\,(3)$	7200
$(1^5)\,(31^2)\,(2^2)\,(21)\,(3)$	360
$(1^5)\,(2^21)\,(31)\,(21)\,(3)$	720
$(21^3)^2\,(31)\,(21)\,(3)$	2400
$(21^3)\,(2^21)\,(1^4)\,(3)^2$	300
$(2^21)\,(1^5)\,(21^2)\,(3)^2$	180
$(21^3)^2\,(21^2)\,(3)^2$	600
$(1^5)\,(21^3)\,(2^2)\,(3)^2$	60
Total	64980

Hence $As\{5^243^2,\,3^22^31^8\} = \dfrac{361}{17280}$.

(*d*) *Combinatorial problem.*

Twenty different books are of five colours, five red, five orange, four yellow, three green and three blue, (P). In how many ways can they be made up into thirteen parcels, two containing three, three containing two and eight containing one book, (Q), so that no two books in the same parcel are of different colours?

(*e*) *Summation properties.*

$$\sum_{Q/\sigma} (p_1!)^{\pi_1} (p_2!)^{\pi_2} \ldots As(P, Q)$$
$$= \text{coefficient of } x^{\sigma} \text{ in } \{x^{(p_1)}\}^{\pi_1} \{x^{(p_2)}\}^{\pi_2} \ldots$$
$$= \text{coefficient of } x^{\sigma} \text{ in } \{x^{(\pi_1')}\}^{p_1'} \{(x-\pi_1')^{(\pi_2')}\}^{p_2'} \ldots,$$

where $\sum_{Q/\sigma}$ denotes summation over all partitions Q of σ parts,

$$x^{(p_1)} \equiv x(x-1)(x-2) \ldots (x-p_1+1),$$
$$x^{(\pi_1')} \equiv x(x-1)(x-2) \ldots (x-\pi_1'+1),$$
$$(x-\pi_1')^{(\pi_2')} \equiv (x-\pi_1')(x-\pi_1'-1) \ldots (x-\pi_1'-\pi_2'+1),$$

and $(p_1'^{\pi_1'} p_2'^{\pi_2'} \ldots)$ is the partition conjugate to $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$.

The bipartitional function $Sg(P, Q)$

(*a*) *Algebraic definition:*

$$s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots = \Sigma Sg(P, Q)\, G(q_1^{\chi_1} q_2^{\chi_2} \ldots),$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ and the summation Σ is taken over all the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$Sg(P, Q)$ is the number of possible two-way distributions such that

(i) The column totals form the parts of the partition P.
(ii) The row totals form the parts of the partition Q.
(iii) Each column is represented in only one row.
(iv) No column with lower total precedes any column with higher total.
(v) No row with lower total precedes any row with higher total.

Example. To evaluate $Sg(2^2 1^4, 3^2 2)$.

There are two separations of P with specification Q satisfying the conditions above, namely,

3	2	.	1	.	.	.
3	.	2	.	1	.	.
2	.	.	.	.	1	1
	2	2	1	1	1	1

3	2	.	.	.	.	1
3	.	.	1	1	1	.
2	.	2	.	.	.	.
	2	2	1	1	1	1

Tables of $(p_1!)^{\pi_1}\,(p_2!)^{\pi_2}\ldots As(P, Q)$. $(p_1!)^{\pi_1}\,(p_2!)^{\pi_2}\ldots a_{p_1}^{\pi_1}a_{p_2}^{\pi_2}\ldots$ in terms of $s_{q_1}^{x_1}s_{q_2}^{x_2}\ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1²)
(2)	−1	1
(1²)	.	1

P \ Q	(3)	(21)	(1³)
(3)	2	−3	1
(21)	.	−1	1
(1³)	.	.	1

P \ Q	(4)	(31)	(2²)	(21²)	(1⁴)
(4)	−6	8	3	−6	1
(31)	.	2	.	−3	1
(2²)	.	.	1	−2	1
(21²)	.	.	.	−1	1
(1⁴)	.	.	.	.	1

P \ Q	(5)	(41)	(32)	(31²)	(2²1)	(21³)	(1⁵)
(5)	24	−30	−20	20	15	−10	1
(41)	.	−6	.	8	3	−6	1
(32)	.	.	−2	2	3	−4	1
(31²)	.	.	.	2	.	−3	1
(2²1)	.	.	.	.	1	−2	1
(21³)	.	.	.	.	.	−1	1
(1⁵)	.	.	.	.	.	.	1

P \ Q	(6)	(51)	(42)	(3²)	(41²)	(321)	(2³)	(31³)	(2²1²)	(21⁴)	(1⁶)
(6)	−120	144	90	40	−90	−120	−15	40	45	−15	1
(51)	.	24	.	.	−30	−20	.	20	15	−10	1
(42)	.	.	6	.	−6	−8	−3	8	9	−7	1
(3²)	.	.	.	4	.	−12	.	4	9	−6	1
(41²)	.	.	.	.	−6	.	.	8	3	−6	1
(321)	.	.	.	.	.	−2	.	2	3	−4	1
(2³)	.	.	.	.	.	.	−1	.	3	−3	1
(31³)	.	.	.	.	.	.	.	2	.	−3	1
(2²1²)	.	.	.	.	.	.	.	.	1	−2	1
(21⁴)	.	.	.	.	.	.	.	.	.	−1	1
(1⁶)	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(7)	(61)	(52)	(43)	(51²)	(421)	(3²1)	(32²)	(41³)	(321²)	(2³1)	(31⁴)	(2²1³)	(21⁵)	(1⁷)
(7)	720	−840	−504	−420	504	630	280	210	−210	−420	−105	70	105	−21	1
(61)	.	−120	.	.	144	90	40	.	−90	−120	−15	40	45	−15	1
(52)	.	.	−24	.	24	30	.	20	−30	−40	−15	20	25	−11	1
(43)	.	.	.	−12	.	18	16	6	−6	−36	−9	10	21	−9	1
(51²)	.	.	.	.	24	.	.	.	−30	−20	.	20	15	−10	1
(421)	.	.	.	.	.	6	.	.	−6	−8	−3	8	9	−7	1
(3²1)	.	.	.	.	.	.	4	.	.	−12	.	4	9	−6	1
(32²)	.	.	.	.	.	.	.	2	.	−4	−3	2	7	−5	1
(41³)	.	.	.	.	.	.	.	.	−6	.	.	8	3	−6	1
(321²)	.	.	.	.	.	.	.	.	.	−2	.	2	3	−4	1
(2³1)	.	.	.	.	.	.	.	.	.	.	−1	.	3	−3	1
(31⁴)	.	.	.	.	.	.	.	.	.	.	.	2	.	−3	1
(2²1³)	.	.	.	.	.	.	.	.	.	.	.	.	1	−2	1
(21⁵)	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	1
(1⁷)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(8)	(71)	(62)	(53)	(4²)	(61²)	(521)	(42²)	(431)
(8)	−5040	5760	3360	2688	1260	−3360	−4032	−1260	−3360
(71)	.	720	.	.	.	−840	−504	.	−420
(62)	.	.	120	.	.	−120	−144	−90	.
(53)	.	.	.	48	.	.	−72	.	−60
(4²)	.	.	.	.	36	.	.	−36	−96
(61²)	.	.	.	.	.	−120	.	.	.
(521)	.	.	.	.	.	.	−24	.	.
(42²)	.	.	.	.	.	.	.	−6	.
(431)	.	.	.	.	.	.	.	.	−12
(3²2)	.	.	.	.	.	.	.	.	.
(51³)	.	.	.	.	.	.	.	.	.
(421²)	.	.	.	.	.	.	.	.	.
(3²1²)	.	.	.	.	.	.	.	.	.
(32²1)	.	.	.	.	.	.	.	.	.
(2⁴)	.	.	.	.	.	.	.	.	.
(41⁴)	.	.	.	.	.	.	.	.	.
(321³)	.	.	.	.	.	.	.	.	.
(2³1²)	.	.	.	.	.	.	.	.	.
(31⁵)	.	.	.	.	.	.	.	.	.
(2²1⁴)	.	.	.	.	.	.	.	.	.
(21⁶)	.	.	.	.	.	.	.	.	.
(1⁸)	.	.	.	.	.	.	.	.	.

P \ Q	(3²2)	(51³)	(421²)	(3²1²)	(32²1)	(2⁴)	(41⁴)	(321³)	(2³1²)	(31⁵)	(2²1⁴)	(21⁶)	(1⁸)
(8)	−1120	1344	2520	1120	1680	105	−420	−1120	−420	112	210	−28	1
(71)	.	504	630	280	210	.	−210	−420	−105	70	105	−21	1
(62)	−40	144	180	40	120	15	−90	−160	−60	40	60	−16	1
(53)	−40	24	90	40	90	.	−30	−100	−45	22	45	−13	1
(4²)	.	.	72	64	48	9	−12	−96	−36	16	42	−12	1
(61²)	.	144	90	40	.	.	−90	−120	−15	40	45	−15	1
(521)	.	24	30	.	20	.	−30	−40	−15	20	25	−11	1
(42²)	.	.	12	.	8	3	−6	−16	−12	8	16	−8	1
(431)	.	.	18	16	6	.	−6	−36	−9	10	21	−9	1
(3²2)	−4	.	.	4	12	.	.	−16	−9	4	15	−7	1
(51³)	.	24	.	.	.	.	−30	−20	.	20	15	−10	1
(421²)	.	.	6	.	.	.	−6	−8	−3	8	9	−7	1
(3²1²)	.	.	.	4	.	.	.	−12	.	4	9	−6	1
(32²1)	.	.	.	.	2	.	.	−4	−3	2	7	−5	1
(2⁴)	.	.	.	.	.	1	.	.	−4	.	6	−4	1
(41⁴)	.	.	.	.	.	.	−6	.	.	8	3	−6	1
(321³)	.	.	.	.	.	.	.	−2	.	2	3	−4	1
(2³1²)	.	.	.	.	.	.	.	.	−1	.	3	−3	1
(31⁵)	.	.	.	.	.	.	.	.	.	2	.	−3	1
(2²1⁴)	.	.	.	.	.	.	.	.	.	.	1	−2	1
(21⁶)	.	.	.	.	.	.	.	.	.	.	.	−1	1
(1⁸)	.	.	.	.	.	.	.	.	.	.	.	.	1

It is easy to see that the first separation will generate twenty-four two-way distributions of its kind by assigning the two twos in the two rows in two ways and by assigning the four units in the three rows in twelve ways. In the second separation the first two rows may be interchanged; the two twos can be assigned in two ways and the four units in four ways giving in all sixteen two-way distributions of its kind. The total number of two-way distributions is thus forty, which is the value of $Sg(2^21^4, 3^22)$.

(*c*) *Practical evaluation.*

All the separations of P whose specification is Q are listed. Each separation is scored with the product of the numbers of permutations of equal parts p^π among the separates of P, multiplied by the numbers of permutations among separates of equal weight. The sum of these scores for all separations gives $Sg(P, Q)$.

Example. To evaluate $Sg(43^22^31^4, 94^23)$.

The partition $(43^22^31^4)$ has eighteen separations with specification (94^23). The score of each separation is written beside it.

		Subtotal	976
$(3^221)\ (4)\ (2^2)\ (1^3)$	24	$(432)\ (2^2)\ (1^4)\ (3)$	12
$(432)\ (31)\ (2^2)\ (1^3)$	48	$(432)\ (21^2)^2\ (3)$	72
$(32^3)\ (31)\ (4)\ (1^3)$	16	$(431^2)\ (2^2)\ (21^2)\ (3)$	72
$(432)\ (31)\ (21^2)\ (21)$	288	$(42^21)\ (31)\ (21^2)\ (3)$	144
$(431^2)\ (31)\ (2^2)\ (21)$	144	$(421^3)\ (31)\ (2^2)\ (3)$	48
$(42^21)\ (31)^2\ (21)$	144	$(32^3)\ (4)\ (1^4)\ (3)$	4
$(32^21^2)\ (4)\ (31)\ (21)$	144	$(32^21^2)\ (4)\ (21^2)\ (3)$	72
$(3^21^3)\ (4)\ (2^2)\ (21)$	24	$(321^4)\ (4)\ (2^2)\ (3)$	12
$(3^221)\ (4)\ (21^2)\ (21)$	144	$(2^31^3)\ (31)\ (4)\ (3)$	16
Subtotal	976	Total	1428

Hence $Sg(43^22^31^4, 94^23) = 1428$.

(*d*) *Combinatorial problem.*

(1) In how many ways can twenty objects of the type $(43^22^31^4)$ be distributed in twenty parcels of the type (94^23) subject to the restriction that no two different type parcels contain objects of the same kind?

Or (2) In how many ways can twenty objects of the type $(43^22^31^4)$ be divided among four different parcels containing nine, four, four and three objects and subject to the restriction that no two parcels contain objects of the same kind?

(*e*) *Summation properties.*

$$\sum_{Q/\sigma} \frac{Sg(P, Q)}{x_1!\, x_2! \ldots} = \text{coefficient of } a^\sigma \prod_{i=1}^{\rho} x_i \text{ in } \Pi(1+ax_i)\, \Pi(1+ax_i x_j)\, \Pi(1+ax_i x_j x_k) \ldots,$$

where $\sum_{Q/\sigma}$ denotes summation over partitions of Q of σ parts.

$$\sum_{Q} \frac{Sg(P, Q)}{x_1!\, x_2! \ldots} = \text{coefficient of } \prod_{1}^{\rho} x_i \text{ in } \Pi(1+x_i)\, \Pi(1+x_i x_j)\, \Pi(1+x_i x_j x_k) \ldots.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

TABLES OF $Sg(P, Q) \div x_1!\, x_2! \ldots$ $s_{p_1}^{\pi_1} s_{p_2}^{\pi_2} \ldots$ IN TERMS OF $x_1!\, x_2! \ldots G(q_1^{x_1} q_2^{x_2} \ldots)$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1^2)
(2)	1	.
(1^2)	1	1

P \ Q	(3)	(21)	(1^3)
(3)	1	.	.
(21)	1	1	.
(1^3)	1	3	1

P \ Q	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	1	.	.	.	.
(31)	1	1	.	.	.
(2^2)	1	.	1	.	.
(21^2)	1	2	1	1	.
(1^4)	1	4	3	6	1

P \ Q	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(5)	1	.	.	.	.	.	.
(41)	1	1	.	.	.	.	.
(32)	1	.	1	.	.	.	.
(31^2)	1	2	1	1	.	.	.
(2^21)	1	1	2	.	1	.	.
(21^3)	1	3	4	3	3	1	.
(1^5)	1	5	10	10	15	10	1

P \ Q	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
(6)	1	.	.	.	.	.	.	.	.	.	.
(51)	1	1	.	.	.	.	.	.	.	.	.
(42)	1	.	1	.	.	.	.	.	.	.	.
(3^2)	1	.	.	1	.	.	.	.	.	.	.
(41^2)	1	2	1	.	1	.	.	.	.	.	.
(321)	1	1	1	1	.	1	.	.	.	.	.
(2^3)	1	.	3	.	.	.	1	.	.	.	.
(31^3)	1	3	3	1	3	3	.	1	.	.	.
(2^21^2)	1	2	3	2	1	4	1	.	1	.	.
(21^4)	1	4	7	4	6	16	3	4	6	1	.
(1^6)	1	6	15	10	15	60	15	20	45	15	1

P \ Q	(7)	(61)	(52)	(43)	(51^2)	(421)	(3^21)	(32^2)	(41^3)	(321^2)	(2^31)	(31^4)	(2^21^3)	(21^5)	(1^7)
(7)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(61)	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(52)	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(43)	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(51^2)	1	2	1	.	1	.	.	.	.	.	.	.	.	.	.
(421)	1	1	1	1	.	1	.	.	.	.	.	.	.	.	.
(3^21)	1	1	.	2	.	.	1	.	.	.	.	.	.	.	.
(32^2)	1	.	2	1	.	.	.	1	.	.	.	.	.	.	.
(41^3)	1	3	3	1	3	3	.	.	1	.	.	.	.	.	.
(321^2)	1	2	2	3	1	2	2	1	.	1	.	.	.	.	.
(2^31)	1	1	3	3	.	3	.	3	.	.	1	.	.	.	.
(31^4)	1	4	6	5	6	12	4	3	4	6	.	1	.	.	.
(2^21^3)	1	3	5	7	3	9	6	7	1	6	3	.	1	.	.
(21^5)	1	5	11	15	10	35	20	25	10	40	15	5	10	1	.
(1^7)	1	7	21	35	21	105	70	105	35	210	105	35	105	21	1

P \ Q	(8)	(71)	(62)	(53)	(4^2)	(61^2)	(521)	(42^2)	(431)	(3^22)	(51^3)	(421^2)	(3^21^2)	(32^21)	(2^4)	(41^4)	(321^3)	(2^31^2)	(31^5)	(2^21^4)	(21^6)	(1^8)
(8)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(71)	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(62)	1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(53)	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(4^2)	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(61^2)	1	2	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(521)	1	1	1	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(42^2)	1	.	2	.	1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(431)	1	1	.	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(3^22)	1	.	1	2	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(51^3)	1	3	3	1		3	3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(421^2)	1	2	2	2	1	1	2	1	2	.	.	1	.	.	.	.	.	.	.	.	.	.
(3^21^2)	1	2	1	2	2	1	.	.	4	1	.	.	1	.	.	.	.	.	.	.	.	.
(32^21)	1	1	2	3	1	.	2	1	1	2	.	.	.	1	.	.	.	.	.	.	.	.
(2^4)	1	.	4	.	3	.	.	6	.	.	.	.	.	.	1	.	.	.	.	.	.	.
(41^4)	1	4	6	4	1	6	12	3	4	.	4	6	.	.	.	1	.	.	.	.	.	.
(321^3)	1	3	4	5	3	3	6	3	9	4	1	3	3	3	.	.	1	.	.	.	.	.
(2^31^2)	1	2	4	6	3	1	6	6	6	6	.	3	.	6	1	.	.	1	.	.	.	.
(31^5)	1	5	10	11	5	10	30	15	25	10	10	30	10	15	.	5	10	.	1	.	.	.
(2^21^4)	1	4	8	12	7	6	20	16	28	20	4	18	12	28	3	1	8	6	.	1	.	.
(21^6)	1	6	16	26	15	15	66	60	90	70	20	105	60	150	15	15	80	45	6	15	1	.
(1^8)	1	8	28	56	35	28	168	210	280	280	56	420	280	840	105	70	560	420	56	210	28	1

THE BIPARTITIONAL FUNCTION $Ag(P, Q)$

(*a*) *Algebraic definition:*

$$a_{p_1}^{\pi_1} a_{p_2}^{\pi_2} \ldots = \Sigma Ag(P, Q)\, G(q_1^{\chi_1} q_2^{\chi_2} \ldots),$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$; Q stands for the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ and the summation Σ is taken over all the monomial symmetric functions represented by the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$Ag(P, Q)$ is the number of possible ways in which we can fill the cells of a $\rho \times \sigma$ lattice such that

(i) The column totals read from left to right form the parts of the partition P in some fixed order;

(ii) The row totals read from top to bottom form the parts of the partition Q in some fixed order;

(iii) The number in each cell does not exceed unity. Viewed in this way it is evident that $Ag(P, Q) = Ag(Q, P)$.

Example. To evaluate $Ag(321^3, 42^2)$.

There are seven patterns fulfilling the conditions, namely:

$$\begin{array}{c|ccccc} 4 & 1 & . & 1 & 1 & 1 \\ 2 & 1 & 1 & . & . & . \\ 2 & 1 & 1 & . & . & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccc} 4 & 1 & 1 & 1 & 1 & . \\ 2 & 1 & 1 & . & . & . \\ 2 & 1 & . & . & . & 1 \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccc} 4 & 1 & 1 & 1 & . & 1 \\ 2 & 1 & 1 & . & . & . \\ 2 & 1 & . & . & 1 & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccc} 4 & 1 & 1 & . & 1 & 1 \\ 2 & 1 & 1 & . & . & . \\ 2 & 1 & . & 1 & . & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{c|ccccc} 4 & 1 & 1 & 1 & 1 & . \\ 2 & 1 & . & . & . & 1 \\ 2 & 1 & 1 & . & . & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccc} 4 & 1 & 1 & 1 & . & 1 \\ 2 & 1 & . & . & 1 & . \\ 2 & 1 & 1 & . & . & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccc} 4 & 1 & 1 & . & 1 & 1 \\ 2 & 1 & . & 1 & . & . \\ 2 & 1 & 1 & . & . & . \\ \hline & 3 & 2 & 1 & 1 & 1 \end{array}$$

Hence $Ag(321^3, 42^2) = 7$.

(*c*) *Practical evaluation for larger partitions.*

It is clear from (*b*) that $Ag(P, Q)$ can in general be evaluated by writing down the possible number of patterns fulfilling the conditions of the definition. As it happens, however, with high-order partitions this number turns out to be very large and requires a careful and systematic arrangement of the patterns, lest some may miss being enumerated.

As would appear from (*b*), the procedure of evaluating $Ag(P, Q)$ is essentially as follows:

Corresponding to the first part of Q, units are picked in all possible ways from the parts of P, one unit being picked from each part of P. From the partitions obtained after the operation of the first part of Q, units corresponding to the second part of Q are picked in all possible ways and the process is continued till the parts of Q are exhausted. Thus in the example above $Ag(321^3, 42^2)$, corresponding to the part 4 of the partition (42^2), we may pick up the four units from the parts of (321^3) in four ways, leaving the partition (21^2) in three cases and the partition (2^2) in one case. Corresponding to the second part 2 of (42^2), the two units are picked from (21^2) in three ways, yielding the partitions (2) and (1^2) in one and two ways respectively; and the two units are picked from (2^2) in one way yielding the partition (1^2). Thus to sum up we have two partitions (2) and (1^2) arrived at in three and seven ways respectively. Operating with the last part of (42^2) namely 2, we obtain the number 7, since we can pick out the units from the partitions (2) and (1^2) in zero and one way respectively. The process is systematically put down as follows:

		2				
		(21^2)	(2^2)		(0)	
4	(321^3)	3	1			
2	(2)	3	.	3	.	
	(1^2)	6	1	7	7	
						7

Alternatively we may operate with the parts of P on the partition Q, arriving at the same result as follows:

		2		*1*			
		(31^2)		(2)	(1^2)		(0)
3	(42^2)	1					
1	(3)	1	1	1	.		
	(21)	2	2	2	2		
				3	2		
1	(1)			3	4	7	7
							7

A larger example will illustrate the facility with which this process may be carried out. The duplicate calculation affords a completely independent check.

Example. To evaluate $Ag\{(94^23), (43^22^31^4)\}$.

		4					*3*			
		(42^21^3)	(3^221^3)	(32^31^2)	(32^21^4)		(3)	(21)	(1^3)	
9	$(43^22^31^4)$	1	2	3	4					
4	(421)	2	.	.	.	2	.	.	.	
	(41^3)	3	.	.	.	3	3	.	.	
	(321^2)	6	12	9	32	59	.	59	.	
	(32^2)	1	4	.	4	9	.	.	.	
	(31^4)	3	.	6	24	33	33	132	.	
	(3^21)	.	2	.	.	2	.	.	.	
	(2^31)	.	6	9	16	31	.	.	31	
	(2^21^3)	.	6	18	48	72	.	144	216	
	(21^5)	.	.	3	16	19	.	95	190	
							36	430	437	
						(0)	.	.	437	437

	3	*2*							*2*					*1*			*1*
	(83^22)	(81^2)	(721)	(71^3)	(631)	(62^2)	(621^2)		(6)	(51)	(42)	(41^2)		(3)	(21)		(1)
4 (94^23)	1																
(82^21)	1	1	2	1	.	.	.										
3 (7321)	2	.	2	.	2	2	2										
(72^3)	1	.	.	1	.	.	3										
		1	4	2	2	2	5										
(8)		1	.	.	.	.	.	1	.	.	.	.					
(71)		2	4	6	.	.	.	12	12	.	.	.					
(62)		.	4	.	2	.	5	11	.	11	.	.					
2 (61^2)		.	4	6	.	2	10	22	22	44	.	.					
(53)		.	.	.	2	.	.	2	.	.	2	.					
(521)		.	.	.	2	4	10	16	.	16	16	16					
(51^3)		.	.	.	.	.	5	5	.	15	.	15					
									34	86	18	31					
(4)									.	86	.	31	117	117	.		
1 (31)									.	.	18	62	80	80	80		
														197	80		
1 (2)														197	80	277	277
(1^2)														.	80	80	160
																	437

(*d*) *Combinatorial problem.*

In how many ways can twenty objects of the type (94^23) be distributed among twenty parcels of the type $(43^22^31^4)$ subject to the restriction that no two similar parcels contain similar objects?

(*e*) *Summation properties.*

(i) $\sum\limits_{Q/\sigma} \dfrac{\sigma!}{x_1!\,x_2!\,\dots} Ag(P, Q)$ = number of ways of distributing objects of the type $(p_1^{\pi_1} p_2^{\pi_2}\dots)$ in parcels of the type (1^σ), subject to the restriction that a parcel does not contain similar objects.

(ii) $$\sum_{Q/\sigma \text{ even}} \frac{\sigma!}{x_1!\,x_2!\,\dots} Ag(P, Q) = \sum_{Q/\sigma \text{ odd}} \frac{\sigma!}{x_1!\,x_2!\,\dots} Ag(P, Q),$$

where the summation $\sum\limits_{Q/\sigma}$ is taken over all partitions of Q of σ parts, and $\sum\limits_{Q/\sigma \text{ even}}$ is taken over all partitions of Q having even number of parts.

It is clear from the symmetry of the table that these properties also hold for columns.

The symmetry of the relationship in such problems between "objects" and "parcels" is perhaps better seen by putting the problem in the different form. Twenty counters are marked on one side, nine with *A*, four with *B*, four with *C* and three with *D*. On the other side they carry numbers: four zeros, three ones, three twos, two threes, two fours, two fives, and one each of six, seven, eight and nine. In how many ways can they be so numbered that all the counters shall be different?

TABLES OF $Ag(P, Q)$. $a_{p_1}^{\pi_1} a_{p_2}^{\pi_2} \ldots$ IN TERMS OF $G(q_1^{\chi_1} q_2^{\chi_2} \ldots)$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1^2)
(2)	.	1
(1^2)	1	2

P \ Q	(3)	(21)	(1^3)
(3)	.	.	1
(21)	.	1	3
(1^3)	1	3	6

P \ Q	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	.	.	.	.	1
(31)	.	.	.	1	4
(2^2)	.	.	1	2	6
(21^2)	.	1	2	5	12
(1^4)	1	4	6	12	24

P \ Q	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(5)	.	.	.	.	.	.	1
(41)	.	.	.	.	.	1	5
(32)	.	.	.	.	1	3	10
(31^2)	.	.	.	1	2	7	20
(2^21)	.	.	1	2	5	12	30
(21^3)	.	1	3	7	12	27	60
(1^5)	1	5	10	20	30	60	120

P \ Q	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
(6)	.	.	.	.	.	.	.	.	.	.	1
(51)	.	.	.	.	.	.	.	.	.	1	6
(42)	.	.	.	.	.	.	.	.	1	4	15
(3^2)	.	.	.	.	.	.	1	.	2	6	20
(41^2)	.	.	.	.	.	.	.	1	2	9	30
(321)	.	.	.	.	.	1	3	3	8	22	60
(2^3)	.	.	.	1	.	3	6	6	15	36	90
(31^3)	.	.	.	.	1	3	6	10	18	48	120
(2^21^2)	.	.	1	2	2	8	15	18	34	78	180
(21^4)	.	1	4	6	9	22	36	48	78	168	360
(1^6)	1	6	15	20	30	60	90	120	180	360	720

P \ Q	(7)	(61)	(52)	(43)	(51^2)	(421)	(3^21)	(32^2)	(41^3)	(321^2)	(2^31)	(31^4)	(2^21^3)	(21^5)	(1^7)
(7)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
(61)	.	.	.	.	.	.	.	.	.	.	.	.	.	1	7
(52)	.	.	.	.	.	.	.	.	.	.	.	.	1	5	21
(43)	.	.	.	.	.	.	.	.	.	.	1	.	3	10	35
(51^2)	.	.	.	.	.	.	.	.	.	.	.	1	2	11	42
(421)	.	.	.	.	.	.	.	.	.	1	3	4	11	35	105
(3^21)	.	.	.	.	.	.	.	1	.	2	7	6	18	50	140
(32^2)	.	.	.	.	.	.	1	2	.	5	12	12	31	80	140
(41^3)	.	.	.	.	.	.	.	.	1	3	6	13	24	75	210
(321^2)	.	.	.	.	.	1	2	5	3	13	27	34	68	170	420
(2^31)	.	.	.	1	.	3	7	12	6	27	51	60	117	270	840
(31^4)	.	.	.	.	1	4	6	12	13	34	60	88	150	360	630
(2^21^3)	.	.	1	3	2	11	18	31	24	68	117	150	258	570	1260
(21^5)	.	1	5	10	11	35	50	80	75	170	270	360	570	1200	2520
(1^7)	1	7	21	35	42	105	140	140	210	420	840	630	1260	2520	5040

P \ Q	(8)	(71)	(62)	(53)	(4^2)	(61^2)	521)	(42^2)	(431)	(3^22)	(51^3)	(421^2)	(3^21^2)	(32^21)	(2^4)	(41^4)	(321^3)	(2^31^2)	(31^5)	(2^21^4)	(21^6)	(1^8)
(8)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
(71)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	8
(62)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	6	28
(53)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	4	15	56
(4^2)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2	.	6	20	70
(61^2)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	2	13	56
(521)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	3	5	14	51	168
(42^2)	.	.	.	.	.	.	.	.	.	.	.	.	1	2	6	.	7	18	20	53	150	420
(431)	.	.	.	.	.	.	.	.	.	.	.	.	.	1	4	.	3	11	10	32	95	280
(3^22)	.	.	.	.	.	.	.	.	.	1	.	.	2	5	12	.	12	31	30	80	210	560
(51^3)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	3	6	16	30	108	336
(421^2)	.	.	.	.	.	.	.	.	.	.	.	1	2	5	12	4	18	39	55	114	315	840
(3^21^2)	.	.	.	.	.	.	.	1	.	2	.	2	4	12	28	6	30	68	80	172	440	1120
(32^21)	.	.	.	.	.	.	.	2	1	5	.	5	12	24	48	12	58	117	140	284	690	1680
(2^4)	.	.	.	.	1	.	.	6	4	12	.	12	28	48	90	24	108	204	240	468	1080	2520
(41^4)	.	.	.	.	.	.	.	.	.	.	1	4	6	12	24	17	46	84	140	246	660	1680
(321^3)	.	.	.	.	.	.	1	7	3	12	3	18	30	58	108	46	141	258	340	612	1440	3360
(2^31^2)	.	.	.	1	2	.	3	18	11	31	6	39	68	117	204	84	258	453	570	1008	2250	5040
(31^5)	.	.	.	.	.	1	5	20	10	30	16	55	80	140	240	140	340	570	800	1320	3000	6720
(2^21^4)	.	.	1	4	6	2	14	53	32	80	30	114	172	284	468	246	612	1008	1320	2172	4680	10080
(21^6)	.	1	6	15	20	13	51	150	95	210	108	315	440	690	1080	660	1440	2250	3000	4680	9720	20160
(1^8)	1	8	28	56	70	56	168	420	280	560	336	840	1120	1680	2520	1680	3360	5040	6720	10080	20160	40320

The bipartitional function $Ah(P, Q) \equiv Ha(P, Q)$

(*a*) *Algebraic definition:*

$$a_{p_1}^{\pi_1} a_{p_2}^{\pi_2} \ldots = \Sigma Ah(P, Q)\, h_{q_1}^{\chi_1} h_{q_2}^{\chi_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{\chi_1} q_2^{\chi_2} \ldots)$ and the summation Σ is taken over all the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$(-1)^{w-\sigma} Ah(P, Q)$ is the number of possible two-way distributions such that

(i) The column totals form the partition P.
(ii) The row totals form the partition Q.
(iii) No column with lower total precedes any column with higher total.
(iv) Each row is represented in only one column.
(v) No row represented in a later column precedes any row represented in an earlier column.
(vi) Rows represented in the same column are arranged in any order, equal rows being undistinguished.

Example. To evaluate $Ah(42^2, 2^21^4)$.

The two-way distributions satisfying the six conditions of the definition are eight as under:

1	1	.	.
1	1	.	.
1	1	.	.
1	1	.	.
2	.	2	.
2	.	.	2
	4	2	2

2	2	.	.
2	2	.	.
1	.	1	.
1	.	1	.
1	.	.	1
1	.	.	1
	4	2	2

2	2	.	.
1	1	.	.
1	1	.	.
2	.	2	.
1	.	.	1
1	.	.	1
	4	2	2

2	2	.	.
1	1	.	.
1	1	.	.
1	.	1	.
1	.	1	.
2	.	.	2
	4	2	2

1	1	.	.
2	2	.	.
1	1	.	.
2	.	2	.
1	.	.	1
1	.	.	1
	4	2	2

1	1	.	.
2	2	.	.
1	1	.	.
1	.	1	.
1	.	1	.
2	.	.	2
	4	2	2

1	1	.	.
1	1	.	.
2	2	.	.
2	.	2	.
1	.	.	1
1	.	.	1
	4	2	2

1	1	.	.
1	1	.	.
2	2	.	.
1	.	1	.
1	.	1	.
2	.	.	2
	4	2	2

Hence $Ah(42^2, 2^21^4) = 8$.

(*c*) *Practical evaluation for larger partitions.*

All the separations of Q whose specification is P are listed. Each separation is scored with the product of the numbers of permutations of parts within each separate multiplied by the numbers of permutations of equal separates. The sum of these scores for all the separations gives $(-1)^{w-\sigma} Ah(P, Q)$.

Example. To evaluate $Ah(4^2 3^2 2^3, 3 2^4 1^9)$.

The partition $(32^4 1^9)$ has eighteen separations with specification $(4^2 3^2 2^3)$. The score of each separation is written beside it.

$(31)\ (2^2)\ (21)^2\ (1^2)^3$	16
$(31)\ (2^2)\ (21)\ (1^3)\ (2)\ (1^2)^2$	48
$(31)\ (2^2)\ (1^3)^2\ (2)^2\ (1^2)$	12
$(31)\ (21^2)\ (21)^2\ (2)\ (1^2)^2$	144
$(31)\ (21^2)\ (21)\ (1^3)\ (2)^2\ (1^2)$	144
$(31)\ (21^2)\ (1^3)^2\ (2)^3$	12
$(31)\ (1^4)\ (21)^2\ (2)^2\ (1^2)$	48
$(31)\ (1^4)\ (21)\ (1^3)\ (2)^3$	16
$(2^2)^2\ (3)\ (1^3)\ (1^2)^3$	2
Subtotal	442

Subtotal	442
$(2^2)\ (21^2)\ (3)\ (21)\ (1^2)^3$	24
$(2^2)\ (21^2)\ (3)\ (1^3)\ (2)\ (1^2)^2$	36
$(2^2)\ (1^4)\ (3)\ (21)\ (2)\ (1^2)^2$	24
$(2^2)\ (1^4)\ (3)\ (1^3)\ (2)^2\ (1^2)$	12
$(21^2)^2\ (3)\ (21)\ (2)\ (1^2)^2$	108
$(21^2)^2\ (3)\ (1^3)\ (2)^2\ (1^2)$	54
$(21^2)\ (1^4)\ (3)\ (21)\ (2)^2\ (1^2)$	72
$(21^2)\ (1^4)\ (3)\ (1^3)\ (2)^3$	12
$(1^4)^2\ (3)\ (21)\ (2)^3$	4
Total	788

(*d*) *Combinatorial problem.*

In how many ways can twenty similar volumes of which one trio, and four pairs, are inseparable be placed in seven distinguished spans of which two will hold four, two three and three two? The order within each span of books with different number of volumes is material.

(*e*) *Summation properties.*

$$\sum_{Q/\sigma} Ah(P, Q) = (-1)^{w-\sigma} \frac{(w-\rho)!}{(w-\sigma)!\,(\sigma-\rho)!},$$

$$\sum_{P/\rho} \frac{\rho!}{\pi_1!\,\pi_2!\,\ldots} Ah(P, Q) = (-1)^{w-\sigma} \frac{\sigma!}{x_1!\,x_2!\,\ldots} \frac{(\sigma-1)!}{(\rho-1)!\,(\sigma-\rho)!},$$

$$\sum_{Q} (-1)^{w-\sigma} Ah(P, Q) = 2^{w-\rho},$$

where $\sum_{Q/\sigma}$ denotes summation over all partitions Q of σ parts,

$\sum_{P/\rho}$ „ „ P of ρ parts, and

$\sum_{Q}$ „ „ Q.

The bipartitional function $Hg(P, Q)$

(*a*) *Algebraic definition:*

$$h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} \ldots = \Sigma Hg(P, Q)\, g(q_1^{x_1} q_2^{x_2} \ldots),$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{x_1} q_2^{x_2} \ldots)$ and the summation Σ is taken over all the monomial symmetric functions represented by the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$Hg(P, Q)$ is the number of possible ways in which we can fill the cells of a $\rho \times \sigma$ lattice such that

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TABLES OF $Ah(P, Q)$. $a_{p_1}^{\pi_1} a_{p_2}^{\pi_2} \ldots$ IN TERMS OF $h_{q_1}^{x_1} h_{b_2}^{x_2} \ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1^2)
(2)	−1	1
(1^2)	.	1

P \ Q	(3)	(21)	(1^3)
(3)	1	−2	1
(21)	.	−1	1
(1^3)	.	.	1

P \ Q	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	−1	2	1	−3	1
(31)	.	1	.	−2	1
(2^2)	.	.	1	−2	1
(21^2)	.	.	.	−1	1
(1^4)	.	.	.	.	1

P \ Q	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(5)	1	−2	−2	3	3	−4	1
(41)	.	−1	.	2	1	−3	1
(32)	.	.	−1	1	2	−3	1
(31^2)	.	.	.	1	.	−2	1
(2^21)	.	.	.	.	1	−2	1
(21^3)	.	.	.	.	.	−1	1
(1^5)	.	.	.	.	.	.	1

P \ Q	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
(6)	−1	2	2	1	−3	−6	−1	4	6	−5	1
(51)	.	1	.	.	−2	−2	.	3	3	−4	1
(42)	.	.	1	.	−1	−2	−1	2	4	−4	1
(3^2)	.	.	.	1	.	−4	.	2	4	−4	1
(41^2)	.	.	.	.	−1	.	.	2	1	−3	1
(321)	.	.	.	.	.	−1	.	1	2	−3	1
(2^3)	.	.	.	.	.	.	−1	.	3	−3	1
(31^3)	.	.	.	.	.	.	.	1	.	−2	1
(2^21^2)	.	.	.	.	.	.	.	.	1	−2	1
(21^4)	.	.	.	.	.	.	.	.	.	−1	1
(1^6)	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(7)	(61)	(52)	(43)	(51^2)	(421)	(3^21)	(32^2)	(41^3)	(321^2)	(2^31)	(31^4)	(2^21^3)	(21^5)	(1^7)
(7)	1	−2	−2	−2	3	6	3	3	−4	−12	−4	5	10	−6	1
(61)	.	−1	.	.	2	2	1	.	−3	−6	−1	4	6	−5	1
(52)	.	.	−1	.	1	2	.	2	−2	−5	−3	3	7	−5	1
(43)	.	.	.	−1	.	2	2	1	−1	−7	−2	3	7	−5	1
(51^2)	.	.	.	.	1	.	.	.	−2	−2	.	3	3	−4	1
(421)	.	.	.	.	.	1	.	.	−1	−2	−1	2	4	−4	1
(3^21)	.	.	.	.	.	.	1	.	.	−4	.	2	4	−4	1
(32^2)	.	.	.	.	.	.	.	1	.	−2	−2	1	5	−4	1
(41^3)	.	.	.	.	.	.	.	.	−1	.	.	2	1	−3	1
(321^2)	.	.	.	.	.	.	.	.	.	−1	.	1	2	−3	1
(2^31)	.	.	.	.	.	.	.	.	.	.	−1	.	3	−3	1
(31^4)	.	.	.	.	.	.	.	.	.	.	.	1	.	−2	1
(2^21^3)	.	.	.	.	.	.	.	.	.	.	.	.	1	−2	1
(21^5)	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	1
(1^7)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

P \ Q	(8)	(71)	(62)	(53)	(4^2)	(61^2)	(521)	(42^2)	(431)	(3^22)	(51^3)	(421^2)	(3^21^2)	(32^21)	(2^4)	(41^4)	(321^3)	(2^31^2)	(31^5)	(2^21^4)	(21^6)	(1^8)
(8)	−1	2	2	2	1	−3	−6	−3	−6	−3	4	12	6	12	1	−5	−20	−10	6	15	−7	1
(71)	.	1	.	.	.	−2	−2	.	−2	.	3	6	3	3	.	−4	−12	−4	5	10	−6	1
(62)	.	.	1	.	.	−1	−2	−2	.	−1	2	5	1	6	1	−3	−10	−7	4	11	−6	1
(53)	.	.	.	1	.	.	−2	.	−2	−2	1	4	3	7	.	−2	−12	−6	4	11	−6	1
(4^2)	.	.	.	.	1	.	.	−2	−4	.	.	6	4	4	1	−2	−12	−6	4	11	−6	1
(61^2)	.	.	.	.	.	−1	.	.	.	.	2	2	1	.	.	−3	−6	−1	4	6	−5	1
(521)	.	.	.	.	.	.	−1	.	.	.	1	2	.	2	.	−2	−5	−3	3	7	−5	1
(42^2)	.	.	.	.	.	.	.	−1	.	.	.	2	.	2	1	−1	−4	−5	2	8	−5	1
(431)	.	.	.	.	.	.	.	.	−1	.	.	2	2	1	.	−1	−7	−2	3	7	−5	1
(3^22)	.	.	.	.	.	.	.	.	.	−1	.	.	1	4	.	.	−6	−4	2	8	−5	1
(51^3)	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	−2	−2	.	3	3	−4	1
(421^2)	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	−1	−2	−1	2	4	−4	1
(3^21^2)	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	−4	.	2	4	−4	1
(32^21)	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	−2	−2	1	5	−4	1
(2^4)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	−4	.	6	−4	1
(41^4)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	.	.	2	1	−3	1
(321^3)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	.	1	2	−3	1
(2^31^2)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	.	3	−3	1
(31^5)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	−2	1
(2^21^4)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	−2	1
(21^6)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−1	1
(1^8)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

(i) The column totals read from left to right form the parts of the partition P in some fixed order.

(ii) The row totals read from top to bottom form the parts of the partition Q in some fixed order.

Example. To evaluate $Hg(521, 3^2 2)$.

There are seventeen patterns fulfilling the conditions of the definition, namely:

```
3 | 3 . .   3 | . 2 1   3 | 3 . .   3 | 2 . 1   3 | 3 . .   3 | 2 1 .   3 | 1 1 1   3 | 2 1 .
3 | . 2 1   3 | 3 . .   3 | 2 . 1   3 | 3 . .   3 | 2 1 .   3 | 3 . .   3 | 2 1 .   3 | 1 1 1
2 | 2 . .   2 | 2 . .   2 | . 2 .   2 | . 2 .   2 | . 1 1   2 | . 1 1   2 | 2 . .   2 | 2 . .
--+------   --+------   --+------   --+------   --+------   --+------   --+------   --+------
  | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1

3 | 1 2 .   3 | 2 . 1   3 | 3 . .   3 | 1 2 .   3 | 3 . .   3 | 1 1 1   3 | 2 1 .   3 | 2 1 .   3 | 2 . 1
3 | 2 . 1   3 | 1 2 .   3 | 1 2 .   3 | 3 . .   3 | 1 1 1   3 | 3 . .   3 | 2 1 .   3 | 2 . 1   3 | 2 1 .
2 | 2 . .   2 | 2 . .   2 | 1 . 1   2 | 1 . 1   2 | 1 1 .   2 | 1 1 .   2 | 1 . 1   2 | 1 1 .   2 | 1 1 .
--+------   --+------   --+------   --+------   --+------   --+------   --+------   --+------   --+------
  | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1     | 5 2 1
```

(*c*) *Practical evaluation.*

It would appear from the definition given above that the process of evaluating $Hg(P, Q)$ is essentially as follows: All the different partitions of $(w-p_1)$, having parts less than or equal to σ are listed. The number of ways in which the parts of each of the different partitions of $(w-p_1)$ are obtained from the parts of Q are then recorded. Proceeding further, the number of ways in which the parts of each of the different partitions of $w-p_1-p_2$ are obtained from the parts of the partitions of $w-p_1$ are recorded and so on till the parts of P are exhausted. The total number thus obtained is $Hg(P, Q)$. The process is systematically carried out as follows:

Example i. To evaluate $Hg(521, 3^2 2)$.

The different partitions of $(8-5)$, i.e. three, are (3), (21) and (1^3). Now the part 3 of the partition (3) may be obtained in two ways from the parts of $(3^2 2)$; the parts 2 and 1 of the partition (21) may be obtained in six ways from the partition $(3^2 2)$, and finally the parts 1, 1 and 1 of the partition (1^3) may be obtained in one way from the partition $(3^2 2)$. The process is written out as

	(3)	(21)	(1^3)
$(3^2 2)$	2	6	1

Next the different partitions of $(8-5-2)$, i.e. of 1, are listed. The part 1 of the partition (1) may be taken out in one way from the partition (3), in two ways from the partition (21) and in three ways from the partition (1^3), giving the results as

	(3)	(21)	(1^3)	
$(3^2 2)$	2	6	1	(0)
(1)	2	12	3	17

Obviously $Hg(P, Q) = 17$.

Alternatively, the process may be exhibited as

	(5)	(41)	(32)	(31^2)	(2^21)		
(521)	1	2	1	1	1		(0)
(2)	1	2	2	1	2	8	8
(1^2)	—	2	1	3	3	9	9
							17

In this process the operand and the operator are interchanged.

(*d*) *Combinatorial problem.*

In how many ways can twenty objects of the type (9432^2) be distributed among twenty parcels of the type $(11,52^2)$.

(*e*) *Summation properties.*

$\sum_{Q/\sigma} \frac{\sigma!}{x_1!\,x_2!\,\ldots} Hg(P, Q) =$ number of ways of distributing objects of the type $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ in parcels of the type (1^σ),

$$\sum_{Q/\sigma \text{ even}} \frac{\sigma!}{x_1!\,x_2!\,\ldots} Hg(P, Q) = \sum_{Q/\sigma \text{ odd}} \frac{\sigma!}{x_1!\,x_2!\,\ldots} Hg(P, Q),$$

where the summation $\sum_{Q/\sigma}$ is taken over all partitions of Q of σ parts and the summation $\sum_{Q/\sigma \text{ even}}$ is taken over all partitions of Q having even number of parts.

It is clear from the symmetry of the table that these properties also hold for columns.

The bipartitional function $Gs(P, Q)$

(*a*) *Algebraic definition:*

$$G(p_1^{\pi_1} p_2^{\pi_2} \ldots) = \Sigma\, Gs(P, Q)\, s_{q_1}^{x_1} s_{q_2}^{x_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{x_1} q_2^{x_2} \ldots)$ and the summation Σ is taken over all the partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

$(-1)^{\rho+\sigma} (x_1!\,x_2!\,\ldots)(\pi_1!\,\pi_2!\,\ldots)\, Gs(P, Q)$ is the number of ways of arranging the entries into rings in each of the σ rows of the two-way distributions, all entries in any row being regarded as different.

The two-way distributions are such that:

(i) The column totals form the parts of the partition P.
(ii) The row totals form the parts of the partition Q.
(iii) Each column is represented in only one row.
(iv) No column with lower total precedes any column with higher total.
(v) No row with lower total precedes any row with higher total.

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Example ii. To evaluate $Hg(9432^2; 11,52^2)$.

		5																							
		(9)	(81)	(72)	(71^2)	(63)	(621)	(61^3)	(54)	(531)	(52^2)	(521^2)	(51^4)	(4^21)	(432)	(431^2)	(42^21)	(421^3)	(3^3)	(3^221)	(32^3)	(3^21^3)	(32^21^2)	(2^41)	
11	(9432^2)	1	4	4	6	2	12	4	1	6	6	12	1	3	12	12	24	8	1	18	12	3	18	5	
	(4)	1	4	4	6	2	12	4	2	6	6	12	1	6	12	12	24	8	.	.	.	.	.	.	122
	(31)	.	4	4	12	4	24	12	2	24	12	36	4	12	48	72	72	32	6	108	36	24	72	.	620
2	(2^2)	.	.	4	.	2	12	.	1	6	18	12	.	3	36	12	72	8	3	54	72	3	54	30	402
	(21^2)	.	.	.	6	.	24	12	.	12	18	72	6	6	36	72	216	96	3	162	144	36	24	120	1365
	(1^4)	.	.	.	.	.	.	4	.	.	.	12	5	.	.	12	24	40	.	18	12	15	90	25	257

	2		
	(2)	(1^2)	
	122	.	
	620	620	
	804	402	
	1365	4095	
	.	1542	
	2911	6659	
(0)	2911	6659	9570

		4																											
		(11)	(10,1)	(92)	(91^2)	(83)	(821)	(81^3)	(74)	(731)	(72^2)	(721^2)	(65)	(641)	(632)	(631^2)	(62^21)	(5^21)	(542)	(53^2)	(541^2)	(5321)	(52^3)	(4^23)	(4^221)	(43^21)	(432^2)	(3^32)	
9	$(11, 52^2)$	1	3	3	3	1	6	1	1	2	3	3	1	2	2	1	3	2	4	.	2	4	2	.	2	.	2	.	
	(7)	1	3	3	3	1	6	1	1	2	3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	27
	(61)	.	3	3	6	1	12	3	1	4	6	9	1	4	4	3	9	.	.	.	.	.	.	.	.	.	.	.	69
	(52)	.	.	3	.	1	6	.	1	2	6	3	2	2	4	1	6	4	8	.	2	8	6	.	.	.	.	.	65
	(43)	.	.	.	.	1	.	.	2	2	.	.	2	4	2	1	.	4	8	.	4	4	.	.	4	.	2	.	40
	(51^2)	.	.	.	3	.	6	3	.	2	3	9	.	2	2	3	9	4	4	.	6	12	6	.	.	.	.	.	74
3	(421)	.	.	.	.	.	6	.	.	2	6	6	.	4	4	2	12	4	16	.	8	16	12	.	16	.	12	.	126
	(3^21)	.	.	.	.	.	.	.	.	2	.	.	.	2	2	2	.	2	4	.	4	8	.	.	4	.	4	.	34
	(32^2)	.	.	.	.	.	.	.	.	.	3	.	.	.	4	.	3	.	8	.	.	8	6	.	4	.	12	.	48
	(41^3)	.	.	.	.	.	.	1	.	.	.	3	.	.	.	1	3	.	.	.	4	4	2	.	4	.	2	.	24
	(321^2)	.	.	.	.	.	.	.	.	.	.	3	.	.	.	2	6	.	.	.	4	16	6	.	8	.	12	.	57
	(2^31)	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	.	.	.	.	4	8	.	2	.	8	.	25

		2					
		(4)	(31)	(2^2)	(21^2)	(1^4)	
		27	.	.	.	.	
		69	69	.	.	.	
		65	65	65	.	.	
		40	80	40	.	.	
		74	148	.	74	.	
		126	252	126	252	.	
		.	136	34	68	.	
		.	96	144	144	.	
		24	72	.	72	24	
		.	171	57	342	57	
		.	.	75	225	25	
		425	1089	541	1177	106	(0)
2	(2)	425	1089	1082	1177	.	3773
	(1^2)	.	1089	541	3531	636	5797
							9570

TABLES OF $Hg(P,Q)$. $h_{p_1}^{\pi_1} h_{p_2}^{\pi_2} \ldots$ IN TERMS OF $G(q_1^{\chi_1} q_2^{\chi_2} \ldots)$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1^2)
(2)	1	1
(1^2)	1	2

P \ Q	(3)	(21)	(1^3)
(3)	1	1	1
(21)	1	2	3
(1^3)	1	3	6

P \ Q	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	1	1	1	1	1
(31)	1	2	2	3	4
(2^2)	1	2	3	4	6
(21^2)	1	3	4	7	12
(1^4)	1	4	6	12	24

P \ Q	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(5)	1	1	1	1	1	1	1
(41)	1	2	2	3	3	4	5
(32)	1	2	3	4	5	7	10
(31^2)	1	3	4	7	8	13	20
(2^21)	1	3	5	8	11	18	30
(21^3)	1	4	7	13	18	33	60
(1^5)	1	5	10	20	30	60	120

P \ Q	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
(6)	1	1	1	1	1	1	1	1	1	1	1
(51)	1	2	2	2	3	3	3	4	4	5	6
(42)	1	2	3	3	4	5	6	7	8	11	15
(3^2)	1	2	3	4	4	6	7	8	10	14	20
(41^2)	1	3	4	4	7	8	9	13	14	21	30
(321)	1	3	5	6	8	12	15	19	24	38	60
(2^3)	1	3	6	7	9	15	21	24	33	54	90
(31^3)	1	4	7	8	13	19	24	34	42	72	120
(2^21^2)	1	4	8	10	14	24	33	42	58	102	180
(21^4)	1	5	11	14	21	38	54	72	102	192	360
(1^6)	1	6	15	20	30	60	90	120	180	360	720

P \ Q	(7)	(61)	(52)	(43)	(51^2)	(421)	(3^21)	(32^2)	(41^3)	(321^2)	(2^31)	(31^4)	(2^21^3)	(21^5)	(1^7)
(7)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(61)	1	2	2	2	3	3	3	3	4	4	4	5	5	6	7
(52)	1	2	3	3	4	5	5	6	7	8	9	11	12	16	21
(43)	1	2	3	4	4	6	7	8	8	11	13	15	18	25	35
(51^2)	1	3	4	4	7	8	8	9	13	14	15	21	22	31	42
(421)	1	3	5	6	8	12	13	16	19	25	30	39	46	70	105
(3^21)	1	3	5	7	8	13	16	19	20	30	37	46	58	90	140
(32^2)	1	3	6	8	9	16	19	25	25	39	51	62	81	130	210
(41^3)	1	4	7	8	13	19	20	25	34	43	51	73	84	135	210
(321^2)	1	4	8	11	14	25	30	39	43	67	87	114	148	250	420
(2^31)	1	4	9	13	15	30	37	51	51	87	120	150	207	360	630
(31^4)	1	5	11	15	21	39	46	62	73	114	150	208	270	480	840
(2^21^3)	1	5	12	18	22	46	58	81	84	148	207	270	378	690	1260
(21^5)	1	6	16	25	31	70	90	130	135	250	360	480	690	1320	2520
(1^7)	1	7	21	35	42	105	140	210	210	420	630	840	1260	2520	5040

P \ Q	(8)	(71)	(62)	(53)	(4^2)	(61^2)	(521)	(42^2)	(431)	(3^22)	(51^3)	(421^2)	(3^21^2)	(32^21)	(2^4)	(41^4)	(321^3)	(2^31^2)	(31^5)	(2^21^4)	(21^6)	(1^8)
(8)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(71)	1	2	2	2	2	3	3	3	3	3	4	4	4	4	4	5	5	5	6	6	7	8
(62)	1	2	3	3	3	5	5	6	5	6	7	8	8	9	10	11	12	13	16	17	22	28
(53)	1	2	3	4	4	4	6	8	7	9	8	9	12	14	16	15	19	22	26	30	41	56
(4^2)	1	2	3	4	5	4	6	9	8	10	8	12	14	16	19	16	22	26	30	36	50	70
(61^2)	1	3	5	4	4	7	8	9	8	9	13	14	14	15	16	21	22	23	31	32	43	56
(521)	1	3	5	6	6	8	12	16	13	17	19	25	26	31	36	39	47	54	71	80	117	168
(42^2)	1	3	6	8	9	9	16	26	20	29	25	40	45	57	72	63	89	111	140	173	270	420
(431)	1	3	5	7	8	8	13	20	17	23	20	31	36	43	52	47	66	80	100	122	185	280
(3^22)	1	3	6	7	10	9	17	29	23	35	26	45	54	69	88	70	108	139	170	220	350	560
(51^3)	1	4	7	8	8	13	19	25	20	26	34	43	44	52	60	73	85	96	136	150	228	336
(421^2)	1	4	8	9	12	14	25	40	31	45	43	68	76	96	120	115	160	198	265	324	525	840
(3^21^2)	1	4	8	12	14	14	26	45	36	54	44	76	92	116	148	126	194	248	320	412	680	1120
(32^21)	1	4	9	14	16	15	31	57	43	69	52	96	116	154	204	162	260	345	440	584	990	1680
(2^4)	1	4	10	16	19	16	36	72	52	88	60	120	148	204	282	204	348	480	600	828	1440	2520
(41^4)	1	5	11	15	16	21	39	63	47	70	73	115	126	162	204	209	286	354	500	606	1020	1680
(321^3)	1	5	12	19	22	22	47	111	66	108	85	160	194	260	348	286	463	618	820	1092	1920	3360
(2^31^2)	1	5	13	22	26	23	54	140	80	139	96	198	248	345	480	354	618	861	1110	1548	2790	5040
(31^5)	1	6	16	26	30	31	71	173	100	170	136	265	320	440	600	500	820	1110	1520	2040	3720	6720
(2^21^4)	1	6	17	30	36	32	80	270	122	220	150	324	412	584	828	606	1092	1548	2040	2892	5400	10080
(21^6)	1	7	22	41	50	43	117	270	185	350	228	525	680	990	1440	1020	1920	2790	3720	5400	10440	20160
(1^8)	1	8	28	56	70	56	168	420	280	560	336	840	1120	1680	2520	1680	3360	5040	6720	10080	20160	40320

Example. To evaluate $(-1)^9\,(1!\,2!)\,(2!\,4!)\,Gs(2^2 1^4, 42^2)$.

There are three separations of the partition $(2^2 1^4)$ with specification (42^2), namely:

$$
\begin{array}{c|cccccc}
4 & 2 & 2 & . & . & . & . \\
2 & . & . & 1 & 1 & . & . \\
2 & . & . & . & . & 1 & 1 \\
\hline
 & 2 & 2 & 1 & 1 & 1 & 1
\end{array}
\qquad
\begin{array}{c|cccccc}
4 & 2 & 1 & 1 & . & . & . \\
2 & . & . & . & 2 & . & . \\
2 & . & . & . & . & 1 & 1 \\
\hline
 & 2 & 1 & 1 & 2 & 1 & 1
\end{array}
\qquad
\begin{array}{c|cccccc}
4 & 1 & 1 & 1 & 1 & . & . \\
2 & . & . & . & . & 2 & . \\
2 & . & . & . & . & . & 2 \\
\hline
 & 1 & 1 & 1 & 1 & 2 & 2
\end{array}
$$

It is easy to see that the first separation will generate six two-way distributions of its kind by assigning the four units in the two rows in six ways; the second separation will generate twenty-four distributions of its kind by assigning the four units in six ways, the two twos in two ways and interchanging the two rows in two ways; the third separation will generate two distributions of its kind by assigning the two twos in two ways. There is only one way of arranging the entries into rings in each of the three rows of the first separation; there are two ways of arranging the entries into rings in the first row and one way each in the other two rows of each of the twenty-four distributions of the second separation; and there are six ways of arranging the entries into rings in the first row and one way each in the other two rows of each of the two distributions of the third separation. Hence we have

$$
\begin{aligned}
(-1)^9\,(1!\,2!)\,(2!\,4!)\,Gs(2^2 1^4, 42^2) &= 6\times 1+24\times 2+2\times 6 \\
&= 66.
\end{aligned}
$$

It is important to notice that $(-1)^{\rho+\sigma}\,(x_1!\,x_2!\,\ldots)\,(\pi_1!\,\pi_2!\,\ldots)\,Gs(P, Q)$ is the number of ways of arranging the entries into rings in each of the σ rows of the two-way distributions of $Sg(P, Q)$, all entries in any row being distinguished.

(*c*) *Practical evaluation.*

All the separations of P whose specification is Q are listed. If in a separation, any separate is made up of v parts of P, then each separation is scored with the product $(v_1-1)!\,(v_2-1)!\ldots$ with the numbers of permutations of equal parts p^π among the separates of P, equal separates (i.e. separates with similar partitions) being undistinguished. The sum of these scores for all separations gives $(-1)^{\rho+\sigma}\,\pi_1!\,\pi_2!\,\ldots\,Gs(P, Q)$.

Example. To evaluate $(-1)^{14}\,(4!)\,(3!)\,(2!)\,Gs(43^2 2^3 1^4, 94^2 3)$.

The partition $(43^2 2^3 1^4)$ has eighteen separations with specification $(94^2 3)$. The score of each separation is written beside it.

		Subtotal	4656
$(3^2 21)\,(4)\,(2^2)\,(1^3)$	144	$(432)\,(2^2)\,(1^4)\,(3)$	72
$(432)\,(31)\,(2^2)\,(1^3)$	96	$(432)\,(21^2)^2\,(3)$	288
$(2^3 3)\,(31)\,(4)\,(1^3)$	96	$(431^2)\,(2^2)\,(21^2)\,(3)$	432
$(432)\,(31)\,(21^2)\,(21)$	576	$(42^2 1)\,(31)\,(21^2)\,(3)$	864
$(431^2)\,(31)\,(2^2)\,(21)$	432	$(421^3)\,(31)\,(2^2)\,(3)$	576
$(42^2 1)\,(31)^2\,(21)$	432	$(32^3)\,(4)\,(1^4)\,(3)$	72
$(32^2 1^2)\,(4)\,(31)\,(21)$	1728	$(32^2 1^2)\,(4)\,(21^2)\,(3)$	1728
$(3^2 1^3)\,(4)\,(2^2)\,(21)$	288	$(321^4)\,(4)\,(2^2)\,(3)$	720
$(3^2 21)\,(4)\,(21^2)\,(21)$	864	$(2^3 1^3)\,(31)\,(4)\,(3)$	960
Subtotal	4656	Total	10368

(*d*) *Combinatorial problem.*

It is proposed that twenty persons belonging to ten different parties of four, three, three, two, two, two, one, one, one, one, should seat themselves around four tables for nine, four, four and three respectively. In how many ways can they do it if it is laid down that

(α) Members of the same party must occupy adjacent seats at the same table, without regard to order within the party.

(β) The order among the different parties is material.

(*e*) *Summation properties.*

$$\sum_{Q/\sigma} \pi_1!\,\pi_2!\,\ldots\, Gs(P, Q) = \text{coefficient of } x^{\sigma} \text{ in } x^{(\rho)}$$
$$= a_{\rho-\sigma},$$

where $\sum_{Q/\sigma}$ denotes summation over partitions of Q of σ parts

$$x^{(\rho)} \equiv x(x-1)\,(x-2)\,\ldots\,(x-\rho+1),$$

and a_1, a_2, $a_3, \ldots$ denote the elementary symmetric functions of the first $\rho-1$ natural numbers.

The bipartitional function $Ga(P, Q)$

(*a*) *Algebraic definition:*

$$G(p_1^{\pi_1} p_2^{\pi_2} \ldots) = \Sigma\, Ga(P, Q)\, a_{q_1}^{x_1} a_{q_2}^{x_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{x_1} q_2^{x_2} \ldots)$ and the summation Σ is taken over all partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

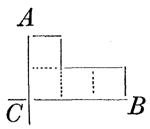
Fill in the cells of a $\rho \times \sigma$ lattice in all different ways such that

(i) The column totals form the parts of the partition P.

(ii) The row totals form the parts of the partition Q.

(iii) A path along the lines of the lattice, starting horizontally from the top left-hand node and ending vertically with the bottom right-hand node divides the lattice into two parts. The cells of the lower portion are all occupied, those of the upper empty.

More clearly, if,



AB is any path starting horizontally from A and ending vertically at B, and we agree to call AB a solid path if every cell in the diagram bounded by the path AB, the vertical AC through A and the horizontal line BC through B is occupied, then condition (iii) is simply stated in an alternative form as follows: It is possible to draw *one* solid path along the lines of the lattice, starting horizontally

Tables of $\pi_1!\,\pi_2!\ldots Gs(P,Q)$; $\pi_1!\,\pi_2!\ldots G(p_1^{\pi_1}p_2^{\pi_2}\ldots)$ in terms of $s_{q_1}^{x_1}s_{q_2}^{x_2}\ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1²)
(2)	1	.
(1²)	−1	1

P \ Q	(3)	(21)	(1³)
(3)	1	.	.
(21)	−1	1	.
(1³)	2	−3	1

P \ Q	(4)	(31)	(2²)	(21²)	(1⁴)
(4)	1	.	.	.	.
(31)	−1	1	.	.	.
(2²)	−1	.	1	.	.
(21²)	2	−2	−1	1	.
(1⁴)	−6	8	3	−6	1

P \ Q	(5)	(41)	(32)	(31²)	(2²1)	(21³)	(1⁵)
(5)	1	.	.	.	.	.	.
(41)	−1	1	.	.	.	.	.
(32)	−1	.	1	.	.	.	.
(31²)	2	−2	−1	1	.	.	.
(2²1)	2	−1	−2	.	1	.	.
(21³)	−6	6	5	−3	−3	1	.
(1⁵)	24	−30	−20	20	15	−10	1

P \ Q	(6)	(51)	(42)	(3²)	(41²)	(321)	(2³)	(31³)	(2²1²)	(21⁴)	(1⁶)
(6)	1	.	.	.	.	.	.	.	.	.	.
(51)	−1	1	.	.	.	.	.	.	.	.	.
(42)	−1	.	1	.	.	.	.	.	.	.	.
(3²)	−1	.	.	1	.	.	.	.	.	.	.
(41²)	2	−2	−1	.	1	.	.	.	.	.	.
(321)	2	−1	−1	−1	.	1	.	.	.	.	.
(2³)	2	.	−3	.	.	.	1	.	.	.	.
(31³)	−6	6	3	2	−3	−3	.	1	.	.	.
(2²1²)	−6	4	5	2	−1	−4	−1	.	1	.	.
(21⁴)	24	−24	−18	−8	12	20	3	−4	−6	1	.
(1⁶)	−120	144	90	40	−90	−120	−15	40	45	−15	1

P \ Q	(7)	(61)	(52)	(43)	(51²)	(421)	(3²1)	(32²)	(41³)	(321²)	(2³1)	(31⁴)	(2²1³)	(21⁵)	(1⁷)
(7)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(61)	−1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(52)	−1	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(43)	−1	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(51²)	2	−2	−1	.	1	.	.	.	.	.	.	.	.	.	.
(421)	2	−1	−1	−1	.	1	.	.	.	.	.	.	.	.	.
(3²1)	2	−1	.	−2	.	.	1	.	.	.	.	.	.	.	.
(32²)	2	.	−2	−1	.	.	.	1	.	.	.	.	.	.	.
(41³)	−6	6	3	2	−3	−3	.	.	1	.	.	.	.	.	.
(321²)	−6	4	3	4	−1	−2	−2	−1	.	1	.	.	.	.	.
(2³1)	−6	2	6	3	.	−3	.	−3	.	.	1	.	.	.	.
(31⁴)	24	−24	−12	−14	12	12	8	3	−4	−6	.	1	.	.	.
(2²1³)	24	−18	−18	−14	6	15	6	8	−1	−6	−3	.	1	.	.
(21⁵)	−120	120	84	70	−60	−90	−40	−35	20	50	15	−5	−10	1	.
(1⁷)	720	−840	−504	−420	504	630	280	210	−210	−420	−105	70	105	−21	1

P \ Q	(8)	(71)	(62)	(53)	(4²)	(61²)	(521)	(42²)	(431)	(3²2)	(51³)	(421²)	(3²1²)	(32²1)	(2⁴)	(41⁴)	(321³)	(2³1²)	(31⁵)	(2²1⁴)	(21⁶)	(1⁸)
(8)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(71)	−1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(62)	−1	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(53)	−1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(4²)	−1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(61²)	2	−2	−1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(521)	2	−1	−1	−1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(42²)	2	.	−2	.	−1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(431)	2	−1	.	−1	−1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(3²2)	2	.	−1	−2	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.
(51³)	−6	6	3	2	.	−3	−3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(421²)	−6	4	3	2	2	−1	−2	−1	−2	.	.	1	.	.	.	.	.	.	.	.	.	.
(3²1²)	−6	4	1	4	2	−1	.	.	−4	−1	.	.	1	.	.	.	.	.	.	.	.	.
(32²1)	−6	2	4	4	1	.	−2	−1	−1	−2	.	.	.	1	.	.	.	.	.	.	.	.
(2⁴)	−6	.	8	.	3	.	.	−6	.	.	.	.	.	.	1	.	.	.	.	.	.	.
(41⁴)	24	−24	−12	−8	−6	12	12	3	8	.	−4	−6	.	.	.	1	.	.	.	.	.	.
(321³)	24	−18	−12	−14	−6	6	9	3	12	5	−1	−3	−3	−3	.	.	1	.	.	.	.	.
(2³1²)	24	−12	−20	−12	−6	2	12	9	6	6	.	−3	.	−6	−1	.	.	1	.	.	.	.
(31⁵)	−120	120	60	64	30	−60	−60	−15	−70	−20	20	30	20	15	.	−5	−10	.	1	.	.	.
(2²1⁴)	−120	96	84	64	30	−36	−72	−33	−56	−28	8	30	12	32	3	−1	−8	−6	.	1	.	.
(21⁶)	720	−720	−480	−384	−180	360	504	180	420	160	−120	−270	−120	−210	−15	30	100	45	−6	−15	1	.
(1⁸)	−5040	5760	3360	2688	1260	−3360	−4032	−1260	−3360	−1120	1344	2520	1120	1680	105	−420	−1120	−420	112	210	−28	1

from the top left-hand node and ending vertically with the bottom right-hand node. A rectangle of occupied cells bounded on two sides by such a path may be thought of as a "step".

If any step of the path of a lattice is made up of c columns and r rows, and the weight of the step is v, then each lattice is scored with the product of

$$\{v_1 \times (c_1-1)! \times (r_1-1)!\}\{v_2 \times (c_2-1)! \times (r_2-1)!\} \ldots,$$

similar rows and columns in any step, which belong to similar compositions of the parts of Q and P, being undistinguished. A positive sign is attached to the score of any lattice if the sum of the number of entries and their total weight is even; a negative sign is attached if it is odd. The sum of these scores for all lattices or patterns is the number sought $Ga(P, Q)$.

Example i. To evaluate $Ga(62, 42^2)$.

There are three different patterns with their respective scores written below them.

$$\begin{array}{c|ccc} 2 & 1 & 1 & \\ 6 & 3 & 1 & 2 \\ \hline & 4 & 2 & 2 \end{array} \qquad \begin{array}{c|ccc} 2 & 1 & 1 & \\ 6 & 1 & 1 & 4 \\ \hline & 2 & 2 & 4 \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & & \\ 6 & 2 & 2 & 2 \\ \hline & 4 & 2 & 2 \end{array}$$

$$-4 \qquad\qquad -4 \qquad\qquad +4 \qquad Ga(62, 42^2) = -4.$$

Note:

$$\begin{array}{c|ccc} 2 & 1 & 1 & \\ 6 & 3 & 1 & 2 \\ \hline & 4 & 2 & 2 \end{array} \qquad \begin{array}{c|ccc} 2 & 1 & 1 & \\ 6 & 1 & 3 & 2 \\ \hline & 2 & 4 & 2 \end{array}$$

these two are not regarded as different patterns, since the parts of P and Q are made up in similar way in either lattice and have the same interrelations.

Example ii. To evaluate $Ga(4^22, 4^22)$:

$$\begin{array}{c|ccc} 2 & 2 & . & . \\ 4 & 1 & 3 & . \\ 4 & 1 & 1 & 2 \\ \hline & 4 & 4 & 2 \end{array} \qquad \begin{array}{c|ccc} 4 & 2 & 2 & . \\ 2 & 1 & 1 & . \\ 4 & 1 & 1 & 2 \\ \hline & 4 & 4 & 2 \end{array} \qquad \begin{array}{c|ccc} 2 & 2 & . & . \\ 4 & 1 & 1 & 2 \\ 4 & 1 & 1 & 2 \\ \hline & 4 & 2 & 4 \end{array} \qquad \begin{array}{c|ccc} 2 & 1 & 1 & . \\ 4 & 2 & 1 & 1 \\ 4 & 1 & 2 & 1 \\ \hline & 4 & 4 & 2 \end{array}$$

$$+12 \qquad\qquad -6 \qquad\qquad -6 \qquad\qquad +2 \qquad Ga(4^22, 4^22) = +2$$

Note: The score of the fourth pattern is two and not four, because of the symmetry in the pattern.

Example iii. To evaluate $Ga(43^2, 43^2)$:

$$\begin{array}{c|ccc} 3 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ \hline & 4 & 3 & 3 \end{array} \qquad \begin{array}{c|ccc} 3 & 2 & 1 & . \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 1 & 2 \\ \hline & 4 & 3 & 3 \end{array}$$

$$-10 \qquad\qquad +9 \qquad Ga(43^2, 43^2) = -1.$$

Example iv. To evaluate $Ga(8543, 74^2 31^2)$:

There are fifty-eight different patterns with their respective scores written below them.

3	1 2	3	1 2	3	1 2	3	1 2	3	1 2 . . . .
4	1 3	4	1 3	4	1 3	4	1 2 1 . . .	4	1 2 1 . . .
5	1 1 1 2 . .	5	1 1 2 1 . .	5	1 1 3 . . .	5	1 2 2 . . .	5	1 1 1 2 . .
8	1 1 3 1 1 1	8	1 1 2 2 1 1	8	1 1 1 3 1 1	8	1 1 1 3 1 1	8	1 2 2 1 1 1
	4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1
	+21		+21		−105		+45		−6

3	1 2	3	1 2	3	1 2	3	1 2	3	1 2 . . . .
4	1 2 1 . . .	4	1 2 1 . . .	4	1 1 2 . . .	4	1 1 2 . . .	4	1 1 2 . . .
5	1 2 1 1 . .	5	1 1 2 1 . .	5	1 1 1 2 . .	5	1 2 1 1 . .	5	1 3 1 . . .
8	1 1 2 2 1 1	8	1 2 1 2 1 1	8	1 3 1 1 1 1	8	1 2 1 2 1 1	8	1 1 1 3 1 1
	4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1
	−3		−3		−12		−6		+45

3	1 2	3	1 2	3	1 2	3	1 2	3	1 2 . . . .
4	1 2 1 . . .	4	1 2 1 . . .	4	1 1 1 1 . .	4	1 1 1 1 . .	5	1 3 1 . . .
5	1 2 1 1 . .	5	1 1 1 2 . .	5	1 1 2 1 . .	5	1 2 1 1 . .	4	1 1 1 1 . .
8	1 1 1 3 1 1	8	1 2 1 2 1 1	8	1 3 1 1 1 1	8	1 2 2 1 1 1	8	1 1 2 2 1 1
	4 7 3 4 1 1		4 7 3 4 1 1		4 7 4 3 1 1		4 7 4 3 1 1		4 7 4 3 1 1
	−3		−6		+15		+12		−3

3	1 2	3	1 2	3	1 1 1 . . .	3	1 1 1 . . .	3	1 1 1 . . .
5	1 3 1 . . .	5	1 2 2 . . .	4	1 1 1 1 . .	4	1 1 2 . . .	4	1 1 2 . . .
4	1 1 1 1 . .	4	1 1 1 1 . .	5	1 1 2 1 . .	5	1 1 1 2 . .	5	1 1 3 . . .
8	1 1 1 3 1 1	8	1 2 1 2 1 1	8	1 1 3 1 1 1	8	1 1 3 1 1 1	8	1 1 1 3 1 1
	4 7 3 4 1 1		4 7 4 3 1 1		4 4 7 3 1 1		4 4 7 3 1 1		4 4 7 3 1 1
	−3		−6		−6		+14		−120

3	1 1 1 . . .	3	1 1 1 . . .	3	3	3	3	3	3 . . . . .
4	1 1 2 . . .	5	1 1 3 . . .	4	2 2	4	2 2	4	2 2 . . . .
5	1 1 2 1 . .	4	1 1 1 1 . .	5	1 1 1 2 . .	5	1 1 2 1 . .	5	1 1 3 . . .
8	1 1 2 2 1 1	8	1 1 2 2 1 1	8	1 1 3 1 1 1	8	1 1 2 2 1 1	8	1 1 1 3 1 1
	4 4 7 3 1 1		4 4 7 3 1 1		7 4 4 3 1 1		7 4 4 3 1 1		7 4 4 3 1 1
	+7		+8		−18		−18		+90

3	3	3	3	3	3	3	3	3	3 . . . . .
4	2 1 1 . . .	4	2 1 1 . . .	4	1 2 1 . . .	4	1 2 1 . . .	4	2 1 1 . . .
5	1 2 2 . . .	5	1 1 1 2 . .	5	2 1 2 . . .	5	1 1 1 2 . .	5	1 1 1 2 . .
8	1 1 1 3 1 1	8	1 2 2 1 1 1	8	1 1 1 3 1 1	8	2 1 2 1 1 1	8	1 2 1 2 1 1
	7 4 4 3 1 1		7 4 4 3 1 1		7 4 4 3 1 1		7 4 4 3 1 1		7 4 3 4 1 1
	−45		+6		−90		+18		+12

3	3	3	3	3	3	3	3	3	3 . . . . .
4	2 1 1 . . .	4	1 2 1 . . .	4	1 2 1 . . .	4	1 1 1 1 . .	4	1 1 1 1 . .
5	1 2 1 1 . .	5	1 1 1 2 . .	5	2 1 1 1 . .	5	1 1 2 1 . .	5	2 1 1 1 . .
8	1 1 1 3 1 1	8	2 1 1 2 1 1	8	1 1 1 3 1 1	8	2 2 1 1 1 1	8	1 2 2 1 1 1
	7 4 3 4 1 1		7 4 3 4 1 1		7 4 3 4 1 1		7 4 4 3 1 1		7 4 4 3 1 1
	+6		+18		+9		−42		−18

3	3	.	.	.	.	.
5	2	2	1	.	.	.
4	1	1	1	1	.	.
8	1	1	2	2	1	1
	7	4	4	3	1	1

+9

3	3	.	.	.	.	.
5	1	2	2	.	.	.
4	1	1	1	1	.	.
8	2	1	1	2	1	1
	7	4	4	3	1	1

+6

3	3	.	.	.	.	.
5	2	2	1	.	.	.
4	1	1	1	1	.	.
8	1	1	1	3	1	1
	7	4	3	4	1	1

+9

3	3	.	.	.	.	.
4	2	1	1	.	.	.
5	1	1	2	1	.	.
8	1	2	1	2	1	1
	7	4	4	3	1	1

+6

3	3	.	.	.	.	.
4	1	2	1	.	.	.
5	2	1	1	1	.	.
8	1	1	2	2	1	1
	7	4	4	3	1	1

+9

3	3	.	.	.	.	.
4	1	1	2	.	.	.
5	1	2	1	1	.	.
8	2	1	1	2	1	1
	7	4	4	3	1	1

+9

4	4	.	.	.	.	.
3	1	2	.	.	.	.
5	1	1	3	.	.	.
8	1	1	1	3	1	1
	7	4	4	3	1	1

+120

4	4	.	.	.	.	.
3	1	2	.	.	.	.
5	1	1	2	1	.	.
8	1	1	2	2	1	1
	7	4	4	3	1	1

−24

4	4	.	.	.	.	.
3	1	2	.	.	.	.
5	1	1	1	2	.	.
8	1	1	3	1	1	1
	7	4	4	3	1	1

−24

4	4	.	.	.	.	
3	1	1	1	.	.	.
5	1	1	1	2	.	.
8	1	2	2	1	1	1
	7	4	4	3	1	1

+8

4	4	.	.	.	.	.
3	1	1	1	.	.	.
5	1	2	2	.	.	.
8	1	1	1	3	1	1
	7	4	4	3	1	1

−60

4	4	.	.	.	.	.
3	1	1	1	.	.	.
5	1	1	1	2	.	.
8	1	1	2	2	1	1
	7	3	4	4	1	1

+16

4	4	.	.	.	.	.
3	1	1	1	.	.	.
5	1	1	2	1	.	.
8	1	1	1	3	1	1
	7	3	4	4	1	1

+8

4	4	.	.	.	.	.
3	1	1	1	.	.	.
5	1	2	1	1	.	.
8	1	1	2	2	1	1
	7	4	4	3	1	1

+8

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	1	1	2	.	.
8	1	2	2	1	1	1
	4	7	4	3	1	1

−8

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	2	2	.	.	.
8	1	1	1	3	1	1
	4	7	4	3	1	1

+60

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	2	1	1	.	.
8	1	1	2	2	1	1
	4	7	4	3	1	1

−4

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	1	2	1	.	.
8	1	2	1	2	1	1
	4	7	4	3	1	1

−4

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	1	1	2	.	.
8	1	2	1	2	1	1
	4	7	3	4	1	1

−8

4	1	3	.	.	.	.
3	1	1	1	.	.	.
5	1	2	1	1	.	.
8	1	1	1	3	1	1
	4	7	3	4	1	1

−4

5	4	1	.	.	.	.
3	1	1	1	.	.	.
4	1	1	1	1	.	.
8	1	1	2	2	1	1
	7	4	4	3	1	1

−5

5	4	1	.	.	.	.
3	1	1	1	.	.	.
4	1	1	2	.	.	.
8	1	1	1	3	1	1
	7	4	4	3	1	1

+75

5	4	1	.	.	.	.
3	1	1	1	.	.	.
4	1	1	1	1	.	.
8	1	1	1	3	1	1
	7	4	3	4	1	1

−5

Total $= 690 - 659 = +31$.
Hence $Ga(8543, 74^2 31^2) = +31$.

(*c*) *Practical evaluation.*

To evaluate $Ga(P, Q)$ we write down the separations of P and Q having common specifications R. Next, corresponding to the partitions R, $Gs(P, R)$ and $Sa(R, Q)$ are evaluated after the manner illustrated before. The sum of the products of the two gives $Ga(P, Q)$. Thus: To evaluate $Ga(8543, 74^2 31^2)$.

	$Gs(8543, R)$		$Sa(R, 74^2 31^2)$	
(8543)	−6	$(74^2 31^2)$	+ 600	−3600
(854) (3)	2	$(74^2 1^2)$ (3)	+ 306	612
(853) (4)	2	$(74^2 1)$ (31)	+ 192	384
		(7431^2) (4)	+ 768	1536
(843) (5)	2	(74^2) (31^2)	+ 75	150
		(7431) (41)	+ 450	900

	$Gs(8543, R)$		$Sa(R, 74^2 31^2)$	
(85) (43)	1	$(4^2 31^2)$ (7)	+ 546	546
		(741^2) (43)	+ 273	273
(543) (8), (84) (53)	3	$(4^2 31)$ (71)	+ 288	864
		(731^2) (4^2)	+ 144	432
		(741) (431)	+ 384	1152
(83) (54)	1	$(4^2 3)$ (71^2)	+ 99	99
		(731) $(4^2 1)$	+ 198	198
		(74) (431^2)	+ 297	297
(8) (43) (5)	−1	(71) (43) (41)	+ 280	− 280
		(4^2) (7) (31^2)	+ 140	− 140
		(431) (7) (41)	+ 560	− 560
(8) (53) (4)	−1	(71) (431) (4)	+1024	−1024
		(71) (4^2) (31)	+ 256	− 256
(54) (8) (3)	−1	$(4^2 1)$ (71) (3)	+ 216	− 216
		(71^2) (4^2) (3)	+ 108	− 108
(83) (5) (4)	−1	(74) (41) (31)	+ 220	− 220
		(731) (41) (4)	+ 440	− 440
		(74) (31^2) (4)	+ 220	− 220
(84) (5) (3)	−1	(741) (41) (3)	+ 360	− 360
(85) (4) (3)	−1	(741^2) (4) (3)	+ 468	− 468
(8) (5) (4) (3)	+1	(71) (41) (4) (3)	+ 480	480
			Total	+ 31

Incidentally, since $Sh(R, Q) \equiv (-1)^{w+\text{number of parts in } R} Sa(R, Q)$, we obtain the value of $Gh(8543, 74^2 31^2)$ by simply making the necessary changes in sign in the entries of the last but one column and therefore in the last column of the table above. In this case it will be 15815. This is a great advantage of this method that we obtain the values of both $Ga(P, Q)$ and $Gh(P, Q)$ with the same amount of work.

(*d*) *Summation properties.*

$$\sum_{Q/\sigma} Ga(P, Q) = (-1)^{w+\rho+\sigma-1} \sum_i \pi_i \times \frac{(\rho-1)!}{\pi_1!\,\pi_2!\,\ldots} \frac{p_i!}{\sigma!\,(p_i-\sigma)!},$$

$$\sum_{P/\rho} \sum_{Q/\sigma} Ga(P, Q) = (-1)^{\rho+\sigma-1} \frac{w!}{(\rho+\sigma-1)!\,(w-\rho-\sigma+1)!},$$

$$\sum_{Q} Ga(P, Q) = (-1)^{w-\rho} \frac{\rho!}{\pi_1!\,\pi_2!\,\ldots},$$

$$\sum_{P} Ga(P, Q) = (-1)^{w-\sigma} \frac{\sigma!}{x_1!\,x_2!\,\ldots},$$

where $\sum_{Q/\sigma}$ denotes summation over all partitions of Q of σ parts; $\sum_{P/\rho}$ denotes summation over partitions of P of ρ parts; $\sum_{Q}$ denotes summation over all the partitions Q and $\sum_{P}$ denotes summation over all the partitions P.

TABLES OF $Ga(P, Q)$. $G(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ IN TERMS OF $a_{q_1}^{x_1} a_{q_2}^{x_2} \ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1^2)
(2)	−2	1
(1^2)	1	.

P \ Q	(3)	(21)	(1^3)
(3)	3	−3	1
(21)	−3	1	.
(1^3)	1	.	.

P \ Q	(4)	(31)	(2^2)	(21^2)	(1^4)
(4)	−4	4	2	−4	1
(31)	4	−1	−2	1	.
(2^2)	2	−2	1	.	.
(21^2)	−4	1	.	.	.
(1^4)	1	.	.	.	.

P \ Q	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
(5)	5	−5	−5	5	5	−5	1
(41)	−5	1	5	−1	−3	1	.
(32)	−5	5	−1	−2	1	.	.
(31^2)	5	−1	−2	1	.	.	.
(2^21)	5	−3	1	.	.	.	.
(21^3)	−5	1	.	.	.	.	.
(1^5)	1	.	.	.	.	.	.

P \ Q	(6)	(51)	(42)	(3^2)	(41^2)	(321)	(2^3)	(31^3)	(2^21^2)	(21^4)	(1^6)
(6)	−6	6	6	3	−6	−12	−2	6	9	−6	1
(51)	6	−1	−6	−3	1	7	2	−1	−4	1	.
(42)	6	−6	2	−3	2	4	−2	−2	1	.	.
(3^2)	3	−3	−3	3	3	−3	1	.	.	.	.
(41^2)	−6	1	2	3	−1	−3	.	1	.	.	.
(321)	−12	7	4	−3	−3	1	.	.	.	.	.
(2^3)	−2	2	−2	1	.	.	.	.	.	.	.
(31^3)	6	−1	−2	.	1	.	.	.	.	.	.
(2^21^2)	9	−4	1	.	.	.	.	.	.	.	.
(21^4)	−6	1	.	.	.	.	.	.	.	.	.
(1^6)	1	.	.	.	.	.	.	.	.		.

P \ Q	(7)	(61)	(52)	(43)	(51^2)	(421)	(3^21)	(32^2)	(41^3)	(321^2)	(2^31)	(31^4)	(2^21^3)	(21^5)	(1^7)
(7)	7	−7	−7	−7	7	14	7	7	−7	−21	−7	7	14	−7	1
(61)	−7	1	7	7	−1	−8	−4	−7	1	9	5	−1	−5	1	.
(52)	−7	7	−3	7	−2	−4	−7	3	2	6	−3	−2	1	.	.
(43)	−7	7	7	−5	−7	−2	5	−1	3	−3	1	.	.	.	.
(51^2)	7	−1	−2	−7	1	3	4	2	−1	−4	.	1	.	.	.
(421)	14	−8	−4	−2	3	8	−1	−2	−3	1	.	.	.	.	.
(3^21)	7	−4	−7	5	4	−1	−2	1	.	.	.	.	.	.	.
(32^2)	7	−7	3	−1	2	−2	1	.	.	.	.	.	.	.	.
(41^3)	−7	1	2	3	−1	−3	.	.	1	.	.	.	.	.	.
(321^2)	−21	9	6	−3	−4	1	.	.	.	.	.	.	.	.	.
(2^31)	−7	5	−3	1	.	.	.	.	.	.	.	.	.	.	.
(31^4)	7	−1	−2	.	1	.	.	.	.	.	.	.	.	.	.
(2^21^3)	14	−5	1	.	.	.	.	.	.	.	.	.	.	.	.
(21^5)	−7	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(1^7)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.

P \ Q	(8)	(71)	(62)	(53)	(4^2)	(61^2)	(521)	(42^2)	(431)	(3^22)	(51^3)	(421^2)	(3^21^2)	(32^21)	(2^4)	(41^4)	(321^3)	(2^31^2)	(31^5)	(2^21^4)	(21^6)	(1^8)
(8)	−8	8	8	8	4	−8	−16	−8	−16	−8	8	24	12	24	2	−8	−32	−16	8	20	−8	1
(71)	8	−1	−8	−8	−4	1	9	8	9	8	−1	−10	−5	−17	−2	1	11	9	−1	−6	1	.
(62)	8	−8	4	−8	−4	2	4	−4	16	2	−2	−6	−9	.	2	2	8	−4	−2	1	.	.
(53)	8	−8	−8	7	−4	8	1	8	1	−7	−3	−9	3	6	−2	3	−3	1	.	.	.	.
(4^2)	4	−4	−4	−4	6	4	8	−4	−8	4	−4	4	2	−4	1	.	.	.	.	.	.	.
(61^2)	−8	1	2	8	4	−1	−3	−2	−9	−5	1	4	5	5	.	−1	−5	.	1	.	.	.
(521)	−16	9	4	1	8	−3	−8	−4	−10	5	3	11	−1	−3	.	−3	1	.	.	.	.	.
(42^2)	−8	8	−4	8	−4	−2	−4	4	.	−2	2	−2	1	.	.	.	.	.	.	.	.	.
(431)	−16	9	16	1	−8	−9	−10	.	10	−1	4	−1	−2	1	.	.	.	.	.	.	.	.
(3^22)	−8	8	2	−7	4	−5	5	−2	−1	1	.	.	.	.	.	.	.	.	.	.	.	.
(51^3)	8	−1	−2	−3	−4	1	3	2	4	.	−1	−4	.	.	.	1	.	.	.	.	.	.
(421^2)	24	−10	−6	−9	4	4	11	−2	−1	.	−4	1	.	.	.	.	.	.	.	.	.	.
(3^21^2)	12	−5	−9	3	2	5	−1	1	−2	.	.	.	.	.	.	.	.	.	.	.	.	.
(32^21)	24	−17	.	6	−4	5	−3	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.
(2^4)	2	−2	2	−2	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(41^4)	−8	1	2	3	.	−1	−3	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.
(321^3)	−32	11	8	−3	.	−5	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2^31^2)	−16	9	−4	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(31^5)	8	−1	−2	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(2^21^4)	20	−6	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(21^6)	−8	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
(1^8)	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

The bipartitional function $Gh(P, Q)$

(*a*) *Algebraic definition:*

$$G(p_1^{\pi_1} p_2^{\pi_2} \ldots) = \Sigma\, Gh(P, Q)\, h_{q_1}^{x_1} h_{q_2}^{x_2} \ldots,$$

where P stands for the partition $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$, Q stands for the partition $(q_1^{x_1} q_2^{x_2} \ldots)$ and the summation Σ is taken over all partitions Q of weight w.

(*b*) *Relation to the enumeration of distributions* in plano.

Fill in the cells of a $\rho \times \sigma$ lattice in all different ways such that

(i) The column totals form the parts of the partition P.
(ii) The row totals form the parts of the partition Q.
(iii) A path along the lines of the lattice, starting horizontally from the top left-hand node and ending vertically with the bottom right-hand node, divides the lattice into two parts, the upper one of which is empty and the lower one is such that the path is composed of one or more independent solid paths in the sense previously defined.

Every solid path of a lattice is scored with a number calculated in exactly the same way as with $Ga(P', Q')$. If the number of entries under the solid path are even, the score is given a negative sign; if odd, it is given a positive sign. The lattice is then scored with the product of these algebraic numbers, multiplied by the number of permutations of solid paths in the lattice, similar paths being undistinguished. The sum of these scores for all lattices gives the number sought, $Gh(P, Q)$.

Example. To evaluate $Gh(7^2 3^2, 4^3 2^2 1^4)$.

There are fifty-two different patterns with their respective scores written below them.

3	1	1	1	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	2	2	2	1	.	.	.	.	.
7	.	.	.	.	2	2	1	1	1
	4	4	4	1	2	2	1	1	1

−28

3	1	2	.	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	2	1	3	1	.	.	.	.	.
7	.	.	.	.	2	2	1	1	1
	4	4	4	1	2	2	1	1	1

+84

3	1	2	.	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	2	1	1	2	1	.	.	.	.
7	.	.	.	.	.	4	1	1	1
	4	4	2	2	1	4	1	1	1

+126

3	1	2	.	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	4	2	1	1	1	4	2	1

+84

3	1	2	.	.	.	.	.	.	.
3	2	1	.	.	.	.	.	.	.
7	1	1	4	1	.	.	.	.	.
7	.	.	.	.	2	2	1	1	1
	4	4	4	1	2	2	1	1	1

−420

3	1	2	.	.	.	.	.	.	.
3	2	1	.	.	.	.	.	.	.
7	1	1	2	2	1	.	.	.	.
7	.	.	.	.	.	4	1	1	1
	4	4	2	2	1	4	1	1	1

−210

3	1	2	.	.	.	.	.	.	.
3	2	1	.	.	.	.	.	.	.
7	1	1	2	1	1	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	4	2	1	1	1	4	2	1

−420

3	1	2	.	.	.	.	.	.	.
7	1	1	3	1	1	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

+30

3	1	1	1	.	.	.	.	.	.
7	1	2	2	1	1	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

−6

3	1	2	.	.	.	.	.	.	.
7	2	1	2	1	1	.	.	.	.
7	1	1	2	1	1	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

+24

3	1	1	1	.	.	.	.	.	.
7	2	2	2	1	.	.	.	.	.
7	1	1	1	1	2	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

+6

3	1	2	.	.	.	.	.	.	.
7	2	1	3	1	.	.	.	.	.
7	1	1	1	1	2	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

−72

3	1	2	.	.	.	.	.	.	.
7	2	1	3	1	.	.	.	.	.
7	1	1	1	1	1	1	1	.	.
3	.	.	.	.	.	.	.	2	1
	4	4	4	2	1	1	1	2	1

−72

3	1	1	1	.	.	.	.	.	.
7	2	2	2	1	.	.	.	.	.
7	1	1	1	1	1	1	1	.	.
3	.	.	.	.	.	.	.	2	1
	4	4	4	2	1	1	1	2	1

+6

3	2	1	.	.	.	.	.	.	.
3	2	1	.	.	.	.	.	.	.
7	.	.	3	3	1	.	.	.	.
7	.	.	1	1	1	1	1	1	1
	4	2	4	4	2	1	1	1	1

+42

3	2	1	.	.	.	.	.	.	.
3	2	1	.	.	.	.	.	.	.
7	.	.	4	2	1	.	.	.	.
7	.	.	.	.	.	4	1	1	1
	4	2	4	2	1	4	1	1	1

+1764

7	2	3	2	.	.	.	.	.	.
7	2	1	2	2	.	.	.	.	.
3	.	.	.	.	2	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

−252

7	2	2	2	1	.	.	.	.	.
7	2	2	2	1	.	.	.	.	.
3	.	.	.	.	2	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

+126

7	2	1	3	1	.	.	.	.	.
7	2	3	1	1	.	.	.	.	.
3	.	.	.	.	2	1	.	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	4	2	2	1	1	1	1

+756

7	2	2	3	.	.	.	.	.	.
7	2	2	1	1	1	.	.	.	.
3	.	.	.	.	.	2	1	.	.
3	.	.	.	.	.	.	.	2	1
	4	4	4	1	1	2	1	2	1

−189

7	2	3	1	1	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	3	.	.
3	.	.	.	.	.	.	1	1	1
	4	4	2	2	1	1	4	1	1

+126

3	2	1	.	.	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	2	.
7	.	.	.	.	.	.	3	2	2
	4	2	1	1	1	1	4	4	2

−36

3	2	1	.	.	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	1	1
7	.	.	.	.	.	.	3	3	1
	4	2	1	1	1	1	4	4	2

+60

3	2	1	.	.	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	3	.	.
7	.	.	.	.	.	.	1	4	2
	4	2	1	1	1	1	4	4	2

+108

3	2	1	.	.	.	.	.	.	.
7	2	1	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	2	.
7	.	.	.	.	.	.	1	2	4
	4	2	1	1	1	1	2	4	4

−72

3	2	1	.	.	.	.	.	.	.
7	2	1	2	1	1	.	.	.	.
3	.	.	.	.	.	3	.	.	.
7	.	.	.	.	.	1	4	1	1
	4	2	2	1	1	4	4	1	1

+432

3	2	1	.	.	.	.	.	.	.
7	2	1	2	1	1	.	.	.	.
3	.	.	.	.	.	2	1	.	.
7	.	.	.	.	.	2	3	1	1
	4	2	2	1	1	4	4	1	1

−72

3	2	1	.	.	.	.	.	.	.
7	2	3	1	1	.	.	.	.	.
7	.	.	.	.	4	2	1	.	.
3	.	.	.	.	.	.	.	2	1
	4	4	1	1	4	2	1	2	1

+756

3	2	1	.	.	.	.	.	.	.
7	2	3	1	1	.	.	.	.	.
3	.	.	.	.	1	1	1	.	.
7	.	.	.	.	3	1	1	1	1
	4	4	1	1	4	2	2	1	1

+18

3	2	1	.	.	.	.	.	.	.
7	2	3	1	1	.	.	.	.	.
3	.	.	.	.	3	.	.	.	.
7	.	.	.	.	1	2	2	1	1
	4	4	1	1	4	2	2	1	1

+162

3	2	1	.	.	.	.	.	.	.
7	2	3	2	.	.	.	.	.	.
3	.	.	.	2	1	.	.	.	.
7	.	.	.	.	.	4	1	1	1
	4	4	2	2	1	4	1	1	1

+756

3	2	1	.	.	.	.	.	.	.
7	2	3	2	.	.	.	.	.	.
3	.	.	.	3	.	.	.	.	.
7	.	.	.	1	2	1	1	1	1
	4	4	2	4	2	1	1	1	1

+216

3	2	1	.	.	.	.	.	.	.
7	2	3	2	.	.	.	.	.	.
3	.	.	.	1	1	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	4	2	1	1	1	4	2	1

+504

3	2	1	.	.	.	.	.	.	.
7	2	1	4	.	.	.	.	.	.
3	.	.	.	2	1	.	.	.	.
7	.	.	.	.	.	4	1	1	1
	4	2	4	2	1	4	1	1	1

+1512

3	2	1	.	.	.	.	.	.	.
7	2	1	4	.	.	.	.	.	.
3	.	.	.	3	.	.	.	.	.
7	.	.	.	1	2	1	1	1	1
	4	2	4	4	2	1	1	1	1

+432

3	2	1	.	.	.	.	.	.	.
7	2	1	4	.	.	.	.	.	.
3	.	.	.	1	1	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	2	4	1	1	1	4	2	1

+1008

3	3	.	.	.	.	.	.	.	.
3	1	2	.	.	.	.	.	.	.
7	.	.	3	3	1	.	.	.	.
7	.	.	1	1	1	1	1	1	1
	4	2	4	4	2	1	1	1	1

−84

3	3	.	.	.	.	.	.	.	.
3	1	2	.	.	.	.	.	.	.
7	.	.	4	1	1	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	2	4	1	1	1	4	2	1

−3528

3	3	.	.	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	.	.	.	3	3	1	.	.	.
7	.	.	.	1	1	1	2	1	1
	4	1	1	4	4	2	2	1	1

−168

3	3	.	.	.	.	.	.	.	.
3	1	1	1	.	.	.	.	.	.
7	.	.	.	4	2	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	1	1	4	2	1	4	2	1

−1764

3	3	.	.	.	.	.	.	.	.
7	1	2	1	1	1	1	.	.	.
3	.	.	.	.	.	.	1	1	1
7	.	.	.	.	.	.	3	3	1
	4	2	1	1	1	1	4	4	2

−360

3	3	.	.	.	.	.	.	.	.
7	1	2	1	1	1	1	.	.	.
3	.	.	.	.	.	.	3	.	.
7	.	.	.	.	.	.	1	4	2
	4	2	1	1	1	1	4	4	2

−648

3	3	.	.	.	.	.	.	.	.
7	1	2	2	1	1	.	.	.	.
3	.	.	.	.	.	3	.	.	.
7	.	.	.	.	.	1	4	1	1
	4	2	2	1	1	4	4	1	1

−972

3	3	.	.	.	.	.	.	.	.
7	1	4	1	1	.	.	.	.	.
3	.	.	.	.	2	1	.	.	.
7	.	.	.	.	.	.	4	2	1
	4	4	1	1	2	1	4	2	1

−4536

$$\begin{array}{c|ccccccccc} 3 & 3 & . & . & . & . & . & . & . & . \\ 7 & 1 & 4 & 1 & 1 & . & . & . & . & . \\ 3 & . & . & . & . & 1 & 1 & 1 & . & . \\ 7 & . & . & . & . & 3 & 1 & 1 & 1 & 1 \\ \hline & 4 & 4 & 1 & 1 & 4 & 2 & 2 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 3 & 3 & . & . & . & . & . & . & . & . \\ 7 & 1 & 4 & 2 & . & . & . & . & . & . \\ 3 & . & . & . & 1 & 2 & . & . & . & . \\ 7 & . & . & . & . & . & 4 & 1 & 1 & 1 \\ \hline & 4 & 4 & 2 & 1 & 2 & 4 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 3 & 3 & . & . & . & . & . & . & . & . \\ 7 & 1 & 4 & 2 & . & . & . & . & . & . \\ 3 & . & . & . & 1 & 1 & 1 & . & . & . \\ 7 & . & . & . & . & . & . & 4 & 2 & 1 \\ \hline & 4 & 4 & 2 & 1 & 1 & 1 & 4 & 2 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 3 & 1 & 1 & 1 & . & . & . & . & . & . \\ 7 & 3 & 3 & 1 & . & . & . & . & . & . \\ 3 & . & . & . & 1 & 2 & . & . & . & . \\ 7 & . & . & . & . & . & 4 & 1 & 1 & 1 \\ \hline & 4 & 4 & 2 & 1 & 2 & 4 & 1 & 1 & 1 \end{array}$$

$-108 \qquad -2268 \qquad -1512 \qquad -1260$

$$\begin{array}{c|ccccccccc} 3 & 1 & 1 & 1 & . & . & . & . & . & . \\ 7 & 3 & 3 & 1 & . & . & . & . & . & . \\ 3 & . & . & . & 1 & 1 & 1 & . & . & . \\ 7 & . & . & . & . & . & . & 4 & 2 & 1 \\ \hline & 4 & 4 & 2 & 1 & 1 & 1 & 4 & 2 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 7 & 3 & 1 & 3 & . & . & . & . & . & . \\ 7 & 1 & 3 & 1 & 2 & . & . & . & . & . \\ 3 & . & . & . & . & 2 & 1 & . & . & . \\ 3 & . & . & . & . & . & . & 1 & 1 & 1 \\ \hline & 4 & 4 & 4 & 2 & 2 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 7 & 3 & 3 & 1 & . & . & . & . & . & . \\ 7 & 1 & 1 & 1 & 4 & . & . & . & . & . \\ 3 & . & . & . & . & 2 & 1 & . & . & . \\ 3 & . & . & . & . & . & . & 1 & 1 & 1 \\ \hline & 4 & 4 & 2 & 4 & 2 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccccccccc} 7 & 3 & 1 & 3 & . & . & . & . & . & . \\ 7 & 1 & 3 & 1 & 1 & 1 & . & . & . & . \\ 3 & . & . & . & . & . & 2 & 1 & . & . \\ 3 & . & . & . & . & . & . & . & 2 & 1 \\ \hline & 4 & 4 & 4 & 1 & 1 & 2 & 1 & 2 & 1 \end{array}$$

$-840 \qquad -252 \qquad -504 \qquad -189$

Therefore $$Gh(7^2 3^2, 4^3 2^2 1^4) = 9138 - 20842$$
$$= -11704.$$

(*c*) *Practical evaluation for larger partitions.*

To evaluate $Gh(P, Q)$ we write down the separations of P and Q having common specifications R. Next, corresponding to the partitions R, $Gs(7^2 3^2, R)$ and $Sh(R, 4^3 2^2 1^4)$ are evaluated after the manner illustrated before. The sum of the products of the two gives $Gh(P, Q)$. Thus: to evaluate $Gh(7^2 3^2, 4^3 2^2 1^4)$:

	$Gs(7^2 3^2, R)$		$Sh(R, 4^3 2^2 1^4)$	
$(7^2 3^2)$	$-\frac{3}{2}$	$(4^3 2^2 1^4)$	$+2800$	-4200
$(7^2 3)\ (3)$	1	$(4^3 2 1^3)\ (21)$	-1020	-1020
		$(4^3 2^2 1)\ (1^3)$	-170	-170
$(7^2)\ (3^2)$	$\frac{1}{4}$	$(4^2 2 1^4)\ (42)$	-1260	-315
		$(4^2 2^2 1^2)\ (41^2)$	-1260	-315
		$(4^3 1^2)\ (2^2 1^2)$	-252	-63
		$(4^3 2)\ (21^4)$	-84	-21
$(73^2)\ (7)$	1	$(4^2 2 1^3)\ (421)$	-1820	-1820
		$(4^2 2^2 1)\ (41^3)$	-546	-546
		$(4^3 1)\ (2^2 1^3)$	-182	-182
$(73)^2$	$\frac{1}{2}$	$(4^2 2)\ (421^4)$	-1000	-500
		$(4^2 1^2)\ (42^2 1^2)$	-1800	-900
$(7)^2\ (3^2)$	$-\frac{1}{4}$	$(421)^2\ (41^2)$	$+1176$	-294
		$(421)\ (41^3)\ (42)$	$+1176$	-294
$(73)\ (7)\ (3)$	-1	$(4^2 1^2)\ (421)\ (21)$	$+630$	-630
		$(4^2 2)\ (421)\ (1^3)$	$+140$	-140
		$(4^2 2)\ (41^3)\ (21)$	$+210$	-210
$(7^2)\ (3)^2$	$-\frac{1}{4}$	$(4^3 1^2)\ (21)^2$	$+252$	-63
		$(4^3 2)\ (21)\ (1^3)$	$+84$	-21
			Total	-11704

(*d*) *Summation properties.*

$$\sum_{Q/\sigma} Gh(P, Q) = (-1)^{\rho+\sigma} \frac{(\rho-1)!}{\pi_1!\,\pi_2!\ldots} \left\{ \rho \times \frac{w!}{\sigma!\,(w-\sigma)!} - \sum_{i=1} \pi_i \times \frac{(w-p_i)!}{\sigma!\,(w-p_i-\sigma)!} \right\},$$

$$\sum_{P/\rho} \sum_{Q/\sigma} Gh(P, Q) = (-1)^{\rho} \{ {}^{w-1}C_{\rho-1} \times (-1)^{\sigma}\, {}^{w}C_{\sigma} + {}^{w-2}C_{\rho-2} \times (-1)^{\sigma+1}\, {}^{w}C_{\sigma+1} + \ldots \},$$

Tables of $Gh(P, Q)$. $G(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ in terms of $h_{q_1}^{\chi_1} h_{q_2}^{\chi_2} \ldots$

P \ Q	(1)
(1)	1

P \ Q	(2)	(1²)
(2)	2	−1
(1²)	−1	1

P \ Q	(3)	(21)	(1³)
(3)	3	−3	1
(21)	−3	5	−2
(1³)	1	−2	1

P \ Q	(4)	(31)	(2²)	(21²)	(1⁴)
(4)	4	−4	−2	4	−1
(31)	−4	7	2	−7	2
(2²)	−2	2	3	−4	1
(21²)	4	−7	−4	10	−3
(1⁴)	−1	2	1	−3	1

P \ Q	(5)	(41)	(32)	(31²)	(2²1)	(21³)	(1⁵)
(5)	5	−5	−5	5	5	−5	1
(41)	−5	9	5	−9	−7	9	−2
(32)	−5	5	11	−8	−11	10	−2
(31²)	5	−9	−8	12	10	−13	3
(2²1)	5	−7	−11	10	14	−14	3
(21³)	−5	9	10	−13	−14	17	−4
(1⁵)	1	−2	−2	3	3	−4	1

P \ Q	(6)	(51)	(42)	(3²)	(41²)	(321)	(2³)	(31³)	(2²1²)	(21⁴)	(1⁶)
(6)	6	−6	−6	−3	6	12	2	−6	−9	6	−1
(51)	−6	11	6	3	−11	−17	−2	11	14	−11	2
(42)	−6	6	14	3	−10	−20	−6	10	19	−12	2
(3²)	−3	3	3	6	−3	−15	−1	6	9	−6	1
(41²)	6	−11	−10	−3	15	21	4	−15	−20	16	−3
(321)	12	−17	−20	−15	21	61	8	−30	−48	34	−6
(2³)	2	−2	−6	−1	4	8	4	−4	−10	6	−1
(31³)	−6	11	10	6	−15	−30	−4	19	26	−21	4
(2²1²)	−9	14	19	9	−20	−48	−10	26	46	−33	6
(21⁴)	6	−11	−12	−6	16	34	6	−21	−33	26	−5
(1⁶)	−1	2	2	1	−3	−6	−1	4	6	−5	1

P \ Q	(7)	(61)	(52)	(43)	(51²)	(421)	(3²1)	(32²)	(41³)	(321²)	(2³1)	(31⁴)	(2²1³)	(21⁵)	(1⁷)
(7)	7	−7	−7	−7	7	14	7	7	−7	−21	−7	7	14	−7	1
(61)	−7	13	7	7	−13	−20	−10	−7	13	33	9	−13	−23	13	−2
(52)	−7	7	17	7	−12	−24	−7	−17	12	36	17	−12	−29	14	−2
(43)	−7	7	7	19	−7	−26	−19	−13	11	45	13	−14	−28	14	−2
(51²)	7	−13	−12	−7	18	25	10	12	−18	−43	−14	18	33	−19	3
(421)	14	−20	−24	−26	25	64	29	30	−33	−101	−36	36	76	−40	6
(3²1)	7	−10	−7	−19	10	29	25	13	−14	−60	−14	20	37	−20	3
(32²)	7	−7	−17	−13	12	30	13	26	−14	−54	−26	17	44	−21	3
(41³)	−7	13	12	11	−18	−33	−14	−14	22	55	18	−23	−43	25	−4
(321²)	−21	33	36	45	−43	−101	−60	−54	55	195	62	−70	−143	78	−12
(2³1)	−7	9	17	13	−14	−36	−14	−26	18	62	30	−21	−54	27	−4
(31⁴)	7	−13	−12	−14	18	36	20	17	−23	−70	−21	28	53	−31	5
(2²1³)	14	−23	−29	−28	33	76	37	44	−43	−143	−54	53	117	−64	10
(21⁵)	−7	13	14	14	−19	−40	−20	−21	25	78	27	−31	−64	37	−6
(1⁷)	1	−2	−2	−2	3	6	3	3	−4	−12	−4	5	10	−6	1

P \ Q	(8)	(71)	(62)	(53)	(4²)	(61²)	(521)	(42²)	(431)	(3²2)	(51³)	(421²)	(3²1²)	(32²1)	(2⁴)	(41⁴)	(321³)	(2³1²)	(31⁵)	(2²1⁴)	(21⁶)	(1⁸)
(8)	8	−8	−8	−8	−4	8	16	8	16	8	−8	−24	−12	−24	−2	8	32	16	−8	−20	8	−1
(71)	−8	15	8	8	4	−15	−23	−8	−23	−8	15	38	19	31	2	−15	−53	−23	15	34	−15	2
(62)	−8	8	20	8	4	−14	−28	−20	−16	−14	14	42	15	48	6	−14	−56	−36	14	41	−16	2
(53)	−8	8	8	23	4	−8	−31	−8	−31	−23	13	39	27	54	2	−13	−67	−31	16	40	−16	2
(4²)	−4	4	4	4	10	−4	−8	−12	−24	−4	4	28	14	20	3	−8	−32	−16	8	20	−8	1
(61²)	8	−15	−14	−8	−4	21	29	14	23	11	−21	−50	−22	−43	−4	21	71	34	−21	−49	22	−3
(521)	16	−23	−28	−31	−8	29	76	28	54	37	−39	−105	−49	−119	−8	39	159	84	−42	−110	46	−6
(42²)	8	−8	−20	−8	−12	14	28	36	32	14	−14	−66	−23	−64	−12	20	80	56	−20	−62	24	−3
(431)	16	−23	−16	−31	−24	23	54	32	98	31	−28	−121	−74	−107	−8	40	176	76	−46	−94	39	−6
(3²2)	8	−8	−14	−23	−4	11	37	14	31	35	−16	−48	−33	−84	−4	16	94	50	−22	−61	24	−3
(51³)	−8	15	14	13	4	−21	−39	−14	−28	−16	26	60	27	58	4	−26	−91	−44	27	64	−29	4
(421²)	−24	38	42	39	28	−50	−105	−66	−121	−48	60	211	99	192	20	−76	−298	−154	82	209	−90	12
(3²1²)	−12	19	15	27	14	−22	−49	−23	−74	−33	27	99	66	102	6	−35	−165	−70	44	103	−45	6
(32²1)	−24	31	48	54	20	−43	−119	−64	−107	−84	58	192	102	258	20	−66	−322	−182	81	228	−93	12
(2⁴)	−2	2	6	2	3	−4	−8	−12	−8	−4	4	20	6	20	5	−6	−24	−20	6	21	−8	1
(41⁴)	8	−15	−14	−13	−8	21	39	20	40	16	−26	−76	−35	−66	−6	31	111	54	−33	−79	36	−5
(321³)	32	−53	−56	−67	−32	71	159	80	176	94	−91	−298	−165	−322	−24	111	489	240	−134	−338	148	−20
(2³1²)	16	−23	−36	−31	−16	34	84	56	76	50	−44	−154	−70	−182	−20	54	240	146	−63	−183	76	−10
(31⁵)	−8	15	14	16	8	−21	−42	−20	−46	−22	27	82	44	81	6	−33	−134	−63	39	94	−43	6
(2²1⁴)	−20	34	41	40	20	−49	−110	−62	−94	−61	64	209	103	228	21	−79	−338	−183	94	251	−110	15
(21⁶)	8	−15	−16	−16	−8	22	46	24	39	24	−29	−90	−45	−93	−8	36	148	76	−43	−110	50	−7
(1⁸)	−1	2	2	2	1	−3	−6	−3	−6	−3	4	12	6	12	1	−5	−20	−10	6	15	−7	1

where $$^{w}C_{\sigma}=\frac{w!}{\sigma!(w-\sigma)!},$$

where $\sum\limits_{Q/\sigma}$ and $\sum\limits_{P/\rho}$ denote summations over all partitions of Q and of P of σ and ρ parts respectively.

I would like to express my thanks to Professor R. A. Fisher, to whom I owe the suggestion that a more systematic study might be possible of the bipartitional functions associated with the theory of symmetric functions than has hitherto been done. I am indebted to him for suggesting to me the methods and notation used in this study, for the application of his methods (mostly unpublished) in the case of several partitional functions, and for the very valuable guidance he has given me throughout in the preparation of this paper.

Summary

The transformation formulae of symmetric functions involve arithmetical functions, each of which depends from two partitions of the same partible number. Of these bipartitional functions twelve may be recognized as fundamental. Some of these have been previously studied individually, and in the present paper an attempt is made to set out systematically their mutual relationships and, with respect to each, their connexions with distributions *in plano* and with the combinatorial problems of which they afford solutions. Particular attention is given to the practical evaluation of these functions for the partitions of numbers up to about twenty, with a view to their use in the simplification of heavy algebraic transformations.

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